

# Globally Hyperbolic Moment System by Generalized Hermite Expansion

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## Abstract

In a recent paper [8], it was revealed that a modified 13-moment system taking intrinsic heat fluxes as variables, instead of the heat fluxes along the coordinate vectors which is adopted in the classical Grad 13-moment system, attains some additional advantages than the classical Grad 13-moment system, particularly including that the equilibrium is turned to be the interior point of its hyperbolicity region. The modified 13-moment system was actually derived from the generalized Hermite expansion of the distribution function, where the anisotropy of Hermite expansion is specified by the full temperature tensor. We extend the method therein in this paper to high order of generalized Hermite expansion to derive arbitrary order moment systems, and proposed a globally hyperbolic regularization to achieve locally well-posedness similar to the method in [4]. Furthermore, the structure of the eigen-system of the coefficient matrix and all characteristic waves are fully clarified. The obtained systems provide a systematic class of hydrodynamic models as the refined version of Euler equations, which is gradually approaching the Boltzmann equation with increasing order of the expansion.

**Keywords:** Hydrodynamic Model; Moment System; Global Hyperbolicity; Regularization; NRxx;

## 1 Introduction

In 1949 [12], Grad proposed the moment expansion method for the Boltzmann equation to derive the macroscopic hydrodynamic systems, as the refined models beyond the Euler equations and the Navier-Stokes-Fourier (NSF) equations. Among the models derived therein, Grad's 13-moment system is one of the most

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well-known models. This system was derived by expanding the distribution function into isotropic Hermite series [11]. Soon after the model proposed, it was found that this model is problematic in a number of aspects, one fatal point of which was that Grad's 13-moment system is not globally hyperbolic. Actually, the hyperbolicity can only be obtained near the equilibrium [21] even for 1D flows. The loss of hyperbolicity directly breaks the local well-posedness of the system, and thus the capability of this model is strictly limited. Historically, Grad's moment system has been included in the textbooks for decades while there are very seldom reports on its success, in spite of the elegant mathematical formation of the system. Aiming on improved well-posedness of Grad's moment system, different efforts have been made both for the 13 moment system and higher order moment system. The approaches may be divided into two folds, including proposing certain dissipation terms derived from the collision term and considering different closure to extend the hyperbolicity region. We refer the regularized Burnett equations [16], regularized 13-moment equations [24, 23], the Pearson-13-moment equations [28], et. al. These methods may alleviate the problem of hyperbolicity to some extent [15, 27, 28].

In a recent study [8], the authors pointed out that the thermodynamic equilibrium is always on the boundary of the hyperbolicity region of Grad's 13-moment system. More precisely, it was proved therein that if an arbitrary small perturbation is applied to the phase density from the equilibrium, the hyperbolicity may break down. This reveals that there does not exist a neighbourhood of the equilibrium such that all the states in this neighbourhood lead to the hyperbolicity of Grad's 13-moment system. Without the hyperbolicity in a neighbourhood of the equilibrium, the well-posedness of the Grad's 13-moment system is not guaranteed even the phase density is extremely close to the equilibrium. This severe drawback may be the possible reason why there are hardly any positive evidences for the Grad's 13-moment system in the last decades. Noticing that the anisotropy plays an essential role in breaking down the hyperbolicity, it was then proposed in [8] a new modified 13-moment model such that the equilibrium state lies in the interior of the hyperbolicity region, even without any hyperbolicity regularization techniques used such as in [4]. This modified system is derived by a generalized Hermite expansion instead of the isotropic Hermite expansion in Grad's method, where the anisotropy is specified by the full temperature tensor. It was found that once the generalized Hermite expansion is adopted, the equilibrium is turned into an interior point of the hyperbolicity region of the full 3D system with 13 moments. It is indicated that the generalized Hermite expansion may be an essential point in further development of high order moment method.

As a macroscopic hydrodynamic model derived from the Boltzmann equation, a necessary requirement is that the model derived has to be invariant under Galilean transformation. To achieve this point, Grad in [12] adopted the Ikenberry type polynomials as the weight functions to retrieve the macroscopic quantities from the distribution function. Actually, the Grad's 13-moment system is the first model obtained beyond classical hydrodynamic system following this way. Since all the components of the temperature tensor are included in the variables of the macroscopic model, it looks inappropriate insisting to expand the distribution function

using the isotropic Hermite polynomials. To study higher order moment method than the 13-moment system proposed in [8], we are motivated to adopt expansion of the distribution function using generalized Hermite polynomials. Particularly, we will propose a regularization to the derived systems following the method in [4] to achieve the globally hyperbolicity, thus the local well-posedness of the regularized system may be attained.

The rest part of this paper is arranged as follows. In section 2, we first derived the moment system based on generalized Hermite expansion to arbitrary order for any dimensional cases. In section 3, the coefficient matrix of the obtained moment system is studied in detail, and we point out that the obtained moment system is lack of global hyperbolicity. In section 4, we propose a globally hyperbolic regularization for the moment system obtained. The eigenvalues and the eigenvectors of the regularized system are explicitly calculated, and the global hyperbolicity of the regularized system is rigorously proved. In section 5, the Riemann problem is investigated. All characteristic waves are either genuinely nonlinear or linearly degenerate, and some properties of the rarefaction waves, the contact waves and the shock waves are investigated. In the appendix, we present the properties and formulas of generalized Hermite polynomials and the proof of some technical results in detail.

Before we start the main text, a conjecture on the distribution of the zeros of Hermite polynomials is presented as below at first: we conjecture that there are no same non-zero zeros of  $He_n(x)$  and  $He_m(x)$  for all  $m, n \in \mathbb{N}$  and  $m \neq n$ . Precisely,

**Conjecture 1.1.** *for any  $m, n \in \mathbb{N}$  and  $m \neq n$ , there are no common non-zero zeros of  $He_n(x)$  and  $He_m(x)$ , -i.e.  $\nexists x \in \mathbb{R} \setminus \{0\}$ , such that  $He_n(x) = He_m(x) = 0$ .*

For the detailed description of this conjecture, please see the appendix of this paper.

## 2 Derivation of Moment System

We consider the Boltzmann equation for kinetic theory of gases as

$$\frac{\partial f}{\partial t} + \sum_{d=1}^D \xi_d \frac{\partial f}{\partial x_d} = Q(f, f), \quad (2.1)$$

where  $f(t, \mathbf{x}, \boldsymbol{\xi})$  is the distribution function and  $(t, \mathbf{x}, \boldsymbol{\xi}) \in \mathbb{R}^+ \times \mathbb{R}^D \times \mathbb{R}^D$ ,  $\mathbf{x} = (x_1, \dots, x_D)$ ,  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_D)$ ,  $D$  is the dimension and  $Q(f, f)$  is the collision term. The macroscopic density  $\rho$ , mean flow velocity  $\mathbf{u} = (u_1, \dots, u_D)$ , pressure tensor  $p_{ij}$  and heat flux  $\mathbf{q} = (q_1, \dots, q_D)$  of the gas along the axis of coordinates are related with the distribution function by, for  $i, j = 1, \dots, D$ ,

$$\begin{aligned} \rho &= \int_{\mathbb{R}^D} f \, d\boldsymbol{\xi}, & \rho u_i &= \int_{\mathbb{R}^D} \xi_i f \, d\boldsymbol{\xi}, \\ p_{ij} &= \int_{\mathbb{R}^D} (\xi_i - u_i)(\xi_j - u_j) f \, d\boldsymbol{\xi}, & q_i &= \frac{1}{2} \int_{\mathbb{R}^D} |\boldsymbol{\xi} - \mathbf{u}|^2 (\xi_i - u_i) f \, d\boldsymbol{\xi}. \end{aligned}$$

The original collision term of the Boltzmann equation is quite complex, and in the present work we only consider the BGK-type collision term as

$$Q(f, f) = \nu(\mathcal{G} - f), \quad (2.2)$$

where  $\nu$  is the collision frequency, and  $\mathcal{G}$  is a certain distribution function depending on the collision model under consideration. For BGK collision model [2],

$$\mathcal{G} = f_M = \frac{\rho}{(2\pi\theta)^{D/2}} \exp\left(-\frac{|\boldsymbol{\xi} - \mathbf{u}|^2}{2\theta}\right), \quad (2.3)$$

where  $\theta = \sum_{d=1}^D \frac{p_{dd}}{D\rho}$  is the macroscopic temperature, and for ES-BGK collision model [14],

$$\mathcal{G} = \frac{\rho}{\sqrt{\det(2\pi\Lambda)}} \exp\left(-\frac{1}{2}(\boldsymbol{\xi} - \mathbf{u})^T \Lambda^{-1}(\boldsymbol{\xi} - \mathbf{u})\right), \quad (2.4)$$

where  $\Lambda_{ij} = bp_{ij}/\rho + (1-b)\theta\delta_{ij}$ ,  $b = 1 - \frac{1}{\text{Pr}} \in [-1/2, 1]$ , and Pr is the Prandtl number which is approximately equal to 2/3 for a monatomic gas. In particular, if Pr = 1, the ES-BGK model degrades into the BGK model.

In 1949, Grad [12] made an Hermite expansion for distribution function  $f$  and obtained the well-known Grad 20 and Grad 13 moment equations. Cai and Li [6] extended it to more general case and obtained a class moment equations of arbitrary order, which is called NRxx method. Below we inherit the basic approach of NRxx, adopt a class of generalized Hermite polynomials, and make a generalized Hermite expansion to derive a class of anisotropic moment system.

Consider the weight function

$$w^{[\Theta]}(\mathbf{v}) = \frac{1}{\sqrt{\det(2\pi\Theta)}} \exp\left(-\frac{1}{2}\mathbf{v}^T \Theta^{-1}\mathbf{v}\right), \quad (2.5)$$

where  $\mathbf{v} = (v_1, \dots, v_D) \in \mathbb{R}^D$  and  $\Theta = (\theta_{ij}) \in \mathbb{R}^{D \times D}$  is a symmetrical positive definite matrix. The generalized Hermite polynomials are defined as

$$He_{\alpha}^{[\Theta]}(\mathbf{v}) = \frac{(-1)^{|\alpha|}}{w^{[\Theta]}(\mathbf{v})} \frac{\partial^{\alpha}}{\partial \mathbf{v}^{\alpha}} w^{[\Theta]}(\mathbf{v}), \quad \alpha \in \mathbb{N}^D, \quad (2.6)$$

where  $\alpha = (\alpha_1, \dots, \alpha_D)$  is a  $D$ -dimensional multi-index,  $\frac{\partial^{\alpha}}{\partial \mathbf{v}^{\alpha}} = \frac{\partial^{|\alpha|}}{\partial v_1^{\alpha_1} \dots \partial v_D^{\alpha_D}}$ , and  $|\alpha| = \alpha_1 + \dots + \alpha_D$ . The difference between the generalized Hermite polynomials and the Hermite polynomials is the anisotropy in the weight function, where the matrix  $\Theta$  in the generalized Hermite polynomials is a scalar in the Hermite polynomials. Then we define the generalized Hermite functions as

$$\mathcal{H}_{\alpha}^{[\Theta]}(\mathbf{v}) = w^{[\Theta]}(\mathbf{v}) He_{\alpha}^{[\Theta]}(\mathbf{v}), \quad \alpha \in \mathbb{N}^D. \quad (2.7)$$

Clearly, the weight function  $w^{[\Theta]}(\mathbf{v})$  is normalized that

$$\int_{\mathbb{R}^D} w^{[\Theta]}(\mathbf{v}) d\mathbf{v} = 1.$$

If any component of  $\alpha$  is negative, we take  $(\cdot)_\alpha$  to be zero for convenience. If  $D = 1$  and  $\Theta = 1$ ,  $\mathcal{H}_\alpha^{[\Theta]}(\mathbf{v})$  degenerate into the Hermite polynomials with Gaussian distribution as weight function (see [1] for details), thus the definitions are consistent to 1D case. The generalized Hermite function is studied in details in the Appendix A, and below we summarize some useful properties of  $\mathcal{H}_\alpha^{[\Theta]}(\mathbf{v})$ :

1. Recursion relation:

$$v_d \mathcal{H}_\alpha^{[\Theta]}(\mathbf{v}) = \sum_{j=1}^D \theta_{jd} \mathcal{H}_{\alpha+e_j}^{[\Theta]}(\mathbf{v}) + \alpha_d \mathcal{H}_{\alpha-e_d}^{[\Theta]};$$

2. Quasi-orthogonality relations:

$$\int_{\mathbb{R}^D} \mathcal{H}_\alpha^{[\Theta]}(\mathbf{v}) \mathcal{H}_\beta^{[\Theta]}(\mathbf{v}) \frac{1}{w^{[\Theta]}} d\mathbf{v} = C_{\alpha,\beta} \delta_{|\alpha|,|\beta|},$$

where  $C_{\alpha,\beta}$  is constant dependent on  $\alpha, \beta$ , and  $\Theta$ .

3. Differential relation:

$$\frac{d\mathcal{H}_\alpha^{[\Theta(t)]}(\mathbf{v}(t))}{dt} = - \sum_{i=1}^D \mathcal{H}_{\alpha+e_i}^{[\Theta(t)]}(\mathbf{v}(t)) \frac{dv_i(t)}{dt} + \frac{1}{2} \sum_{i,j=1}^D \mathcal{H}_{\alpha+e_i+e_j}^{[\Theta(t)]}(\mathbf{v}(t)) \frac{d\theta_{ij}(t)}{dt}, \quad (2.8)$$

where  $e_i$ ,  $i = 1, \dots, D$  is the  $D$ -dimensional unit multi-index with its  $i$ -th entry equal to 1.

We expand the distribution function  $f(t, \mathbf{x}, \boldsymbol{\xi})$  into the series of  $\mathcal{H}_\alpha^{[\Theta]}(\mathbf{v})$  as

$$f(t, \mathbf{x}, \boldsymbol{\xi}) = \sum_{\alpha \in \mathbb{N}^D} f_\alpha(t, \mathbf{x}) \mathcal{H}_\alpha^{[\Theta]}(\boldsymbol{\xi} - \mathbf{u}). \quad (2.9)$$

Substituting the expansion (2.9) into the Boltzmann equation (2.1), and comparing the coefficient of  $\mathcal{H}_\alpha^{[\Theta]}(\boldsymbol{\xi} - \mathbf{u})$ , we obtain

$$\begin{aligned} & \frac{Df_\alpha}{Dt} + \sum_{d,k=1}^D \left( \theta_{dk} \frac{\partial f_{\alpha-e_k}}{\partial x_d} + (\alpha_k + 1) \delta_{kd} \frac{\partial f_{\alpha+e_k}}{\partial x_d} \right) + \\ & \sum_{i=1}^D f_{\alpha-e_i} \frac{Du_i}{Dt} + \sum_{i,d,k=1}^D (\theta_{dk} f_{\alpha-e_i-e_k} + (\alpha_k + 1) \delta_{kd} f_{\alpha-e_i+e_k}) \frac{\partial u_i}{\partial x_d} \\ & \sum_{i,j=1}^D \frac{f_{\alpha-e_i-e_j}}{2} \frac{D\theta_{ij}}{Dt} + \sum_{i,j,d,k=1}^D \frac{1}{2} (\theta_{kd} f_{\alpha-e_i-e_j-e_k} + (\alpha_k + 1) \delta_{kd} f_{\alpha-e_i-e_j+e_k}) \frac{\partial \theta_{ij}}{\partial x_d} \\ & = \nu(\mathcal{G}_\alpha - f_\alpha), \end{aligned} \quad (2.10)$$

where  $\frac{D}{Dt}$  is the material derivation standing for

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{d=1}^D u_d \frac{\partial}{\partial x_d},$$

and  $\mathcal{G}$  is expanded as

$$\mathcal{G} = \sum_{\alpha \in \mathbb{N}^D} \mathcal{G}_\alpha \mathcal{H}_\alpha^{[\Theta]}(\boldsymbol{\xi} - \mathbf{u}).$$

In particular, if we let  $\Theta = \theta \mathbf{I}$ , then the moment system (2.10) is exactly the same as the system derived in [7] without hyperbolic regularization. In this paper hereafter, we let  $\theta_{ij} = p_{ij}/\rho$ . The expansion (2.9) together with the quasi-orthogonality relation of  $\mathcal{H}_\alpha^{[\Theta]}(\mathbf{v})$  yields

$$f_0 = \rho, \quad f_{e_i} = 0, \quad f_{e_i+e_j} = 0, \quad q_i = 2f_{3e_i} + \sum_{d=1}^D f_{e_i+2e_d}, \quad i, j = 1, \dots, D. \quad (2.11)$$

Direct calculations give us

$$\mathcal{G}_0 = \rho, \quad \mathcal{G}_{e_i+e_j} = \frac{1-b}{1+\delta_{ij}}(p\delta_{ij} - p_{ij}), \quad i, j = 1, \dots, D, \quad \mathcal{G}_\alpha = 0, \quad \text{if } |\alpha| \text{ is odd.} \quad (2.12)$$

In particular, in case of  $\alpha = \mathbf{0}$  and noticing  $f_{e_i} = 0$  for  $i = 1, \dots, D$ , we deduce the continuity equation from (2.10) as

$$\frac{D\rho}{Dt} + \rho \sum_{d=1}^D \frac{\partial u_d}{\partial x_d} = 0. \quad (2.13)$$

By setting  $\alpha = e_i$  with  $i = 1, \dots, D$  in (2.10), using (2.11), we obtain the equation of momentum conservation as

$$\rho \frac{Du_i}{Dt} + \sum_{d=1}^D \left( \theta_{id} \frac{\partial \rho}{\partial x_d} + \rho \frac{\partial \theta_{id}}{\partial x_d} \right) = 0. \quad (2.14)$$

By setting  $\alpha = e_i + e_j$  with  $i, j = 1, \dots, D$  and  $i \geq j$  in (2.10), using (2.11), we have the conservation laws of pressure tensor as

$$\frac{2-\delta_{ij}}{2} \rho \frac{D\theta_{ij}}{Dt} + \sum_{d=1}^D \left[ (1+\delta_{id}+\delta_{jd}) \frac{\partial f_{e_i+e_j+e_d}}{\partial x_d} + \rho \theta_{id} \frac{\partial u_j}{\partial x_d} + \rho \theta_{jd} \frac{\partial u_i}{\partial x_d} (1-\delta_{ij}) \right] = \nu \mathcal{G}_{e_i+e_j}. \quad (2.15)$$

For the case  $D = 3$ , if we let  $f_\alpha = 0$ ,  $|\alpha| = 0$ , then (2.13), (2.14) and (2.15) are the well-known 10-moment system [13].

Substituting (2.14) and (2.15) into (2.10) to eliminate the material derivation of  $u_i$  and  $\theta_{ij}$ ,  $i, j = 1, \dots, D$ , we get the governing equation of  $f_\alpha$  as

$$\begin{aligned}
& \frac{Df_\alpha}{Dt} + \sum_{d,k=1}^D \theta_{dk} \frac{\partial f_{\alpha-e_k}}{\partial x_d} + \sum_{d=1}^D (\alpha_d + 1) \frac{\partial f_{\alpha+e_d}}{\partial x_d} \\
& - \sum_{i,d=1}^D f_{\alpha-e_i} \left( \frac{\theta_{id}}{\rho} \frac{\partial \rho}{\partial x_d} + \frac{\partial \theta_{id}}{\partial x_d} \right) + \sum_{i,d=1}^D (\alpha_d + 1) f_{\alpha-e_i+e_d} \frac{\partial u_i}{\partial x_d} \\
& + \frac{1}{2} \sum_{i,j,d,k=1}^D (\theta_{kd} f_{\alpha-e_i-e_j-e_k} + \delta_{kd} (\alpha_k + 1) f_{\alpha-e_i-e_j+e_k}) \frac{\partial \theta_{ij}}{\partial x_d} \\
& - \sum_{i,j,d=1}^D \frac{1 + \delta_{id} + \delta_{jd}}{2 - \delta_{ij}} \frac{f_{\alpha-e_i-e_j}}{\rho} \frac{\partial f_{\alpha+e_i+e_j+e_d}}{\partial x_d} = \nu \left( \mathcal{G}_\alpha - f_\alpha + \sum_{i,j=1}^D \frac{\mathcal{G}_{e_i+e_j}}{\rho} f_{\alpha-e_i-e_j} \right).
\end{aligned} \tag{2.16}$$

Then (2.13), (2.14), (2.15) and (2.16) constitute a moment system with infinite equations, which is a quasi-linear system.

To attain a system with finite number of equations, a truncation has to be applied. Due to the quasi-orthogonality of the basis functions  $\mathcal{H}_\alpha^{[\Theta]}(\mathbf{v})$ , we let  $M \in \mathbb{N}$ ,  $M \geq 2$ , and adopt the finite set of closure coefficients  $\{f_\alpha\}_{|\alpha| \leq M}$ , and discard all the equations with  $Df_\alpha/Dt$  with  $|\alpha| > M$ . Then we get a moment system with finite equations. However, the system obtained is not closed yet since in the equations with  $Df_\alpha/Dt$ ,  $|\alpha| = M$ , the terms of  $f_{\alpha+e_d}$ ,  $d = 1, \dots, D$  are involved. The simplest way to close the system is to inherit Grad's idea [12] to let  $f_\alpha = 0$  with  $|\alpha| = M + 1$  in the moment system. Here we first use Grad's way to close the system, which results in a generalized Grad-type moment system.

We remark here that for  $D = 1$ , the moment system above is the same as the NRxx method in 1D case [6], thus the NRxx method, which is the isotropic Grad-type moment system, may be regarded as a special case of the system obtained here.

### 3 Analysis of Moment System

As has been pointed out in [21], the moment system in [12] is not globally hyperbolic. In [5], the authors showed that the NRxx is also not globally hyperbolic even with  $D = 1$ . In this section, we will point out that the generalized Grad-type moment system shares the same problem. For this purpose, we focus on the properties the coefficient matrix of the generalized Grad-type moment system, and prove that the moment system is not globally hyperbolic.

At first, we reformulate the generalized Grad-type moment system obtained in

the previous section as below. Equations (2.14) and (2.15) are as

$$\frac{Du_i}{Dt} + \sum_{d=1}^D \frac{1}{\rho} \frac{\partial p_{id}}{\partial x_d} = 0, \quad (3.1)$$

$$\begin{aligned} \frac{Dp_{ij}}{Dt} + \sum_{d=1}^D \left( p_{ij} \frac{\partial u_d}{\partial x_d} + p_{id} \frac{\partial u_j}{\partial x_d} + p_{jd} \frac{\partial u_i}{\partial x_d} + (e_i + e_j + e_d)! \cdot \frac{\partial f_{e_i+e_j+e_d}}{\partial x_d} \right) \\ = (1 + \delta_{ij}) \nu \mathcal{G}_{e_i+e_j}, \end{aligned} \quad (3.2)$$

where  $\alpha!$  is defined as  $\alpha! = \prod_{d=1}^D \alpha_d!$  and  $i, j = 1, \dots, D$ . Since  $p = \frac{1}{D} \sum_{i=1}^D p_{ii}$ , we have

$$\frac{Dp}{Dt} + \sum_{i,d=1}^D \frac{2}{D} p_{id} \frac{\partial u_i}{\partial x_d} + \sum_{d=1}^D \left( p \frac{\partial u_d}{\partial x_d} + \frac{2}{D} \frac{\partial q_d}{\partial x_d} \right) = 0, \quad (3.3)$$

where  $\sum_{i=1}^D \mathcal{G}_{2e_i} = 0$  is used. Since

$$\frac{\partial \theta_{ij}}{\partial x_d} = \frac{1}{\rho} \frac{\partial p_{ij}}{\partial x_d} - \frac{\theta_{ij}}{\rho} \frac{\partial \rho}{\partial x_d}$$

holds for any  $i, j, d = 1, \dots, D$ , the moment equations (2.16) is reformulated as

$$\begin{aligned} \frac{Df_\alpha}{Dt} + \sum_{d,k=1}^D \theta_{dk} \frac{\partial f_{\alpha-e_k}}{\partial x_d} + \sum_{d=1}^D (\alpha_d + 1) \frac{\partial f_{\alpha+e_d}}{\partial x_d} \\ + \sum_{i,j,d=1}^D \frac{C_{ijd}(\alpha)}{2\rho} \left( \frac{\partial p_{ij}}{\partial x_d} - \theta_{ij} \frac{\partial \rho}{\partial x_d} \right) + \sum_{i,d=1}^D (\alpha_d + 1) f_{\alpha-e_i+e_d} \frac{\partial u_i}{\partial x_d} \\ - \sum_{i,d=1}^D \frac{f_{\alpha-e_i}}{\rho} \frac{\partial p_{id}}{\partial x_d} - \sum_{i,j,d=1}^D \frac{(e_i + e_j + e_d)!}{2} \frac{f_{\alpha-e_i-e_j}}{\rho} \frac{\partial f_{e_i+e_j+e_d}}{\partial x_d} \\ = \nu \left( \mathcal{G}_\alpha - f_\alpha + \sum_{i,j=1}^D \frac{\mathcal{G}_{e_i+e_j}}{\rho} f_{\alpha-e_i-e_j} \right), \end{aligned} \quad (3.4)$$

where  $C_{ijd}(\alpha)$  is

$$C_{ijd}(\alpha) = \sum_{k=1}^D \theta_{kd} f_{\alpha-e_i-e_j-e_k} + (\alpha_d + 1) f_{\alpha-e_i-e_j+e_d}. \quad (3.5)$$

For later usage, some conventional notations are introduced as follows.

For a vector  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ , we denote  $\mathbf{a}(i:j) = (a_i, \dots, a_j)$ ;

For a matrix  $\mathbf{A} = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$ , we denote

$$\mathbf{A}(i, j:k) = (a_{i,j}, \dots, a_{i,k}), \quad \mathbf{A}(i, :) = \mathbf{A}(i, 1:n),$$

$$\mathbf{A}(i:l, j:k) = \begin{pmatrix} a_{i,j} & a_{i,j+1} & \cdots & a_{i,k} \\ a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{l,j} & a_{l,j+1} & \cdots & a_{l,k} \end{pmatrix},$$



Let

$$\mathcal{S}_{D,M} = \{\alpha \in \mathbb{N}^D \mid |\alpha| \leq M\},$$

we permute the elements of  $\mathcal{S}_{D,M}$  by lexicographic order. Then for any  $\alpha \in \mathcal{S}_{D,M}$ ,

$$\mathcal{N}_D(\alpha) = \sum_{i=1}^D \binom{\sum_{k=D-i+1}^D \alpha_k + i - 1}{i} + 1 \quad (3.6)$$

holds, where  $\mathcal{N}_D(\alpha)$  is the ordinal number of  $\alpha$  in  $\mathcal{S}_{D,M}$ . Noticing  $Me_D$  is the last element of  $\mathcal{S}_{D,M}$ , the cardinal number of set  $\mathcal{S}_{D,M}$  is

$$N = \mathcal{N}_D(Me_D) = \binom{M+D}{D},$$

which is total number of variables in the truncated moment system if a truncation with  $|\alpha| \leq M$  is considered.

With the notations above, we collect the variables in the truncated moment system to form a vector  $\mathbf{w} \in \mathbb{R}^N$  as

$$\begin{aligned} w_1 &= \rho, & w_{i+1} &= u_i, \\ w_{\mathcal{N}_D(e_i+e_j)} &= p_{ij}/(1+\delta_{ij}), & w_{\mathcal{N}_D(\alpha)} &= f_\alpha, \text{ else } \alpha. \end{aligned}$$

where  $i, j = 1, \dots, D$ , and  $|\alpha| \leq M$ . Fig. 3(a) shows the permutation of entries  $\mathbf{w}$  as the variables of the truncation moment system. Collecting together (2.13), (3.1), (3.2) and (3.4), we arrive the following quasi-linear system

$$\frac{D\mathbf{w}}{Dt} + \sum_{d=1}^D \mathbf{A}_M^{(d)} \frac{\partial \mathbf{w}}{\partial x_d} = \nu \mathbf{Q} \mathbf{w}, \quad (3.7)$$

where the entries of  $\mathbf{A}_M^{(d)}$  with  $d = 1, \dots, D$  and  $\mathbf{Q}$  are given in (2.13), (3.1), (3.2) and (3.4). The matrices  $\mathbf{A}_M^{(d)}$  have quite regular structure, though complex. Next we will devote to study in detail these coefficient matrix  $\mathbf{A}_M^{(d)}$ .

### 3.1 Properties of the coefficient matrix

Without loss of generality, we only investigate  $\mathbf{A}_M^{(1)}$ . For simplicity, we momentarily strip away the subscripts and use  $\mathbf{A}_M$  to replace  $\mathbf{A}_M^{(1)}$  without ambiguity.

### 3.1.1 Case $D = 1$

This case has been thoroughly studied in [5]. Let us recall the results therein below for comparison. In this case, the coefficient matrix is precisely as

$$B_M = \begin{pmatrix} U & \rho & 0 & \dots\dots\dots & 0 \\ 0 & U & 2\rho^{-1} & 0 & \dots\dots\dots & 0 \\ 0 & 3p_{11}/2 & U & 3 & 0 & \dots\dots\dots & 0 \\ -1/2\theta_{11}^2 & 4f_3 & \theta_{11} & U & 4 & 0 & \dots\dots\dots & 0 \\ -\frac{5\theta_{11}f_3}{2\rho} & 5f_4 & 3\frac{f_3}{\rho} & \theta_{11} & U & 5 & 0 & \dots\dots\dots & 0 \\ -3\frac{\theta_{11}f_4}{\rho} & 6f_5 & 4\frac{f_4}{\rho} & \frac{-3f_3}{\rho} & \theta_{11} & U & 6 & 0 & \dots & 0 \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ -\frac{M\theta_{11}f_{M-2}+\theta_{11}^2f_{M-4}}{2\rho} & Mf_{M-1} & \frac{(M-2)f_{M-2}+\theta_{11}f_{M-4}}{\rho} & \frac{-3f_{M-3}}{\rho} & 0 & \dots & 0 & \theta_{11} & U & M \\ -\frac{(M+1)\theta_{11}f_{M-1}+\theta_{11}^2f_{M-3}}{2\rho} & (M+1)f_M & \frac{(M-1)f_{M-1}+\theta_{11}f_{M-3}}{\rho} & \frac{-3f_{M-2}}{\rho} & 0 & \dots\dots & 0 & \theta_{11} & U \end{pmatrix}, \quad (3.8)$$

where  $U = 0$ . It is clear that this matrix

- is independent of  $u$  and the diagonal entries are all zeros;
- is a lower Hessenberg matrix.

The characteristic polynomial of this matrix is

$$\theta_{11}^{M+1} He_{M+1}^{[\theta_{11}]}(\lambda) - (M+1)! \left( \lambda f_M + \frac{\lambda^2 - \theta_{11}}{2} f_{M-1} \right). \quad (3.9)$$

If  $f_M$  and  $f_{M-1}$  are taken certain values, the characteristic polynomial may not have  $M+1$  real roots, thus the matrix can not be diagonalizable with real eigenvalues. The eigenvector of this matrix for the eigenvalue  $\lambda$ , satisfying (3.9) is

$$r_1 = \rho, r_2 = \lambda, r_3 = \frac{\rho\lambda^2}{2}, r_k = \frac{\rho He_{k-1}^{[\theta_{11}]}(\lambda)}{(k-1)!} - f_{k-2}\lambda - f_{k-3} \frac{He_2^{[\theta_{11}]}(\lambda)}{2}, k = 4, \dots, M+1. \quad (3.10)$$

### 3.1.2 Case $D \geq 2$

We are interested in the case  $D \geq 2$ . Let us investigate some examples at first for a full clarification of the structure of the coefficient matrix.

**Example 1.** If  $D = 2$ , the ordinal number of  $\alpha$  in  $\mathcal{S}_{D,M}$  is  $\mathcal{N}_D(\alpha) = \frac{(\alpha_1 + \alpha_2 + 1)(\alpha_1 + \alpha_2)}{2} + \alpha_2 + 1$ . The permutation of entries of  $\mathbf{w}$  is showed in Fig. 3(a). For the simple

case, the matrix  $\mathbf{A}_3$  is

$$\mathbf{A}_3 = \begin{pmatrix} \begin{array}{cccccc|cccc} 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\rho^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 3p_{11}/2 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 2p_{12} & p_{11} & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & p_{22}/2 & p_{12} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \\ -\theta_{11}^2/2 & 4f_{30} & 0 & \theta_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ -3\theta_{11}\theta_{12}/2 & 3f_{21} & 3f_{30} & \theta_{12} & \theta_{11} & 0 & 0 & 0 & 0 & 0 \\ -\theta_{11}\theta_{22}/2 - \theta_{12}^2 & 2f_{12} & 2f_{21} & 0 & \theta_{12} & \theta_{11} & 0 & 0 & 0 & 0 \\ -\theta_{22}\theta_{12}/2 & f_{03} & f_{12} & 0 & 0 & \theta_{12} & 0 & 0 & 0 & 0 \end{pmatrix},$$

where  $f_{ij} = f_{ie_1+je_2}$ , and the upper-left part in the box is denoted by  $\mathbf{A}_2$ . If  $M > 3$ , for any  $\alpha \in \mathbb{N}^2$ , and  $3 < |\alpha| \leq M$ , we have

$$\mathbf{A}_M(1:10, 1:10) = \mathbf{A}_3, \quad (3.11a)$$

$$\mathbf{A}_M(\mathcal{N}_D(\alpha), \mathcal{N}_D(\alpha)) = 0, \quad (3.11b)$$

$$\mathbf{A}_M(\mathcal{N}_D(\alpha), \mathcal{N}_D(\alpha - e_k)) = \theta_{1k}, \quad \text{if } \alpha_k > 0, \quad (3.11c)$$

$$\mathbf{A}_M(\mathcal{N}_D(\alpha), \mathcal{N}_D(\alpha + e_1)) = \alpha_1 + 1, \quad \text{if } |\alpha| < M, \quad (3.11d)$$

$$\begin{aligned} \mathbf{A}_M(\mathcal{N}_D(\alpha), 1:9) = & \left( -\sum_{i,j=1}^D \frac{\theta_{ij}C_{ij1}}{2\rho}, (\alpha_1 + 1)f_\alpha, (\alpha_1 + 1)f_{\alpha+e_1-e_2}, \right. \\ & \frac{C_{111}(\alpha)}{\rho} - \frac{2f_{\alpha-e_1}}{\rho}, \frac{C_{121}(\alpha)}{\rho} - \frac{f_{\alpha-e_2}}{\rho}, \frac{C_{221}(\alpha)}{\rho}, \\ & \left. -\frac{3f_{\alpha-2e_1}}{\rho}, -\frac{2f_{\alpha-e_1-e_2}}{\rho}, -\frac{f_{\alpha-2e_2}}{\rho} \right), \end{aligned} \quad (3.11e)$$

where  $C_{ijd}(\alpha)$  are given in (3.5). We remark that

- any entry of  $\mathbf{A}_M(i, j)$ , if not specified above, is taken as zero;
- for  $|\alpha| = 4$ , some entries  $\mathbf{A}_M(i, j)$  may be doubly defined in (3.11c) and (3.11e), the value of which is the sum of the both expression;
- if any entries of  $\alpha$  is negative,  $(\cdot)_\alpha$  is taken as zero.

Clearly the matrix  $\mathbf{A}_M$  is independent of  $\mathbf{u}$ , and the diagonal entries vanish. Actually, in (2.13), (3.1), (3.2) and (3.4), the coefficients of terms with derivative to  $x_d$ ,  $d = 1, \dots, D$ , are independent of  $\mathbf{u}$ ; and in the equation containing  $\frac{Dw_i}{Dt}$ ,  $i = 1, \dots, N$ , the coefficients of  $\frac{\partial w_i}{\partial x_d}$ ,  $d = 1, \dots, D$  are zero. Hence, we have that

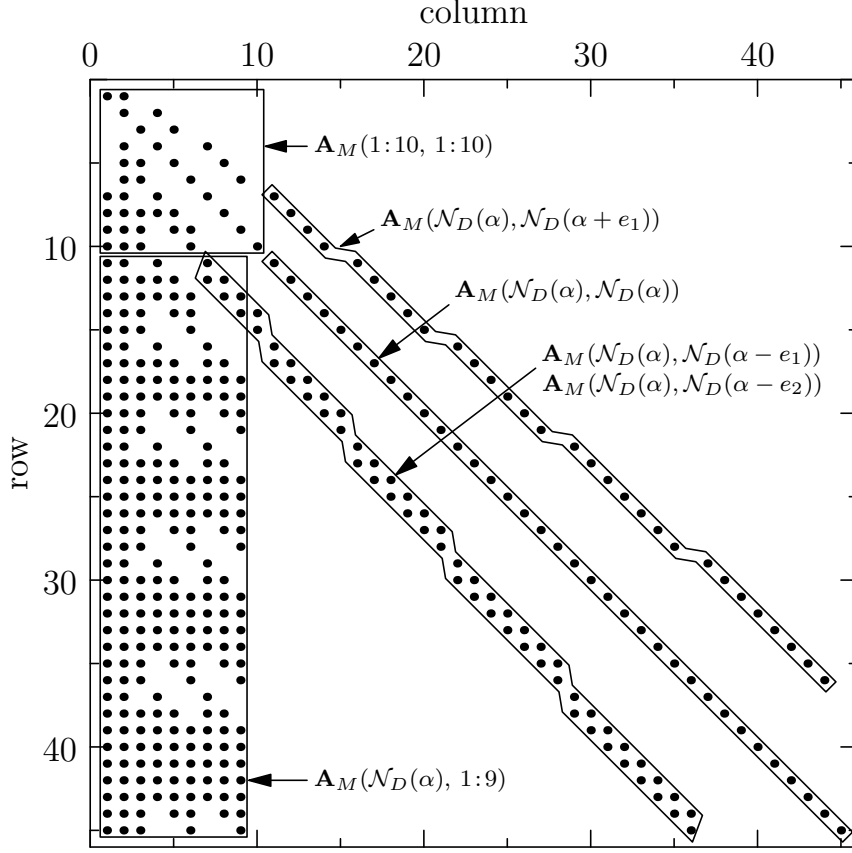


Figure 1: The sparsity pattern of  $\mathbf{A}_M$  with  $M = 8, D = 2$ . Its nonzero entries are given in (3.11).

**Property 1.** *The coefficient matrix  $\mathbf{A}$  is independent of  $\mathbf{u}$ , thus*

$$\frac{\partial \mathbf{A}}{\partial \mathbf{u}} = 0,$$

*and the diagonal entries of  $\mathbf{A}$  are all zeros.*

By this property, the moment system is invariant under a Galilean translation.

In example 1, the coefficient matrix  $\mathbf{A}_M$  for  $D = 2$  is explicitly given, which makes one able to study the sparsity pattern of  $\mathbf{A}_M$ . Fig. 1 gives the sparsity pattern of  $\mathbf{A}_M$  with  $D = 2$  and  $M = 8$ . It is clear that there are at most one nonzero entry in  $\mathbf{A}_M(i, i+1 : N)$ ,  $i = 1, \dots, N$ . Actually, in the equation containing  $\frac{Df_\alpha}{Dt}$  in (3.4), the only nonzero entry is  $\mathbf{A}_M(\mathcal{N}_D(\alpha), \mathcal{N}_D(\alpha + e_1))$  in  $\mathbf{A}_M(\mathcal{N}_D(\alpha), \mathcal{N}_D(\alpha) + 1 : N)$ . Thus, we have the following property.

**Property 2.** *For each  $\alpha \in \mathbb{N}^D$ ,  $|\alpha| \leq M$ , let  $i = \mathcal{N}_D(\alpha)$ , then there are no more than one entry of  $\mathbf{A}_M(i, i+1 : N)$  to be nonzero. In particular, for  $D = 1$ ,  $\mathbf{A}_M$  is a lower Hessenberg matrix.*

Property 2 provides us the approach to calculate the eigenvalues and eigenvectors of  $\mathbf{A}_M$ , the same as operating on a lower Hessenberg matrix. Furthermore, its lower triangular part is quite sparse. Let us try to illustrate its sparsity pattern below.

**Example 2.** Let  $D = 2$ . Considering only the coefficient matrix  $\mathbf{A}_M = \mathbf{A}_M^{(1)}$ , we assume that  $\frac{\partial \cdot}{\partial x_2} = 0$  here. (2.13) shows the  $\frac{D\rho}{Dt}$  is dependent on  $\frac{\partial u_1}{\partial x_1}$ , and we denote the dependence by

$$\rho \rightarrow u_1.$$

Then dependency relationship of entries in  $\mathbf{w}$  by the equations (2.13), (3.1), (3.2) and (3.4) is demonstrated by the graph in Fig. 2.

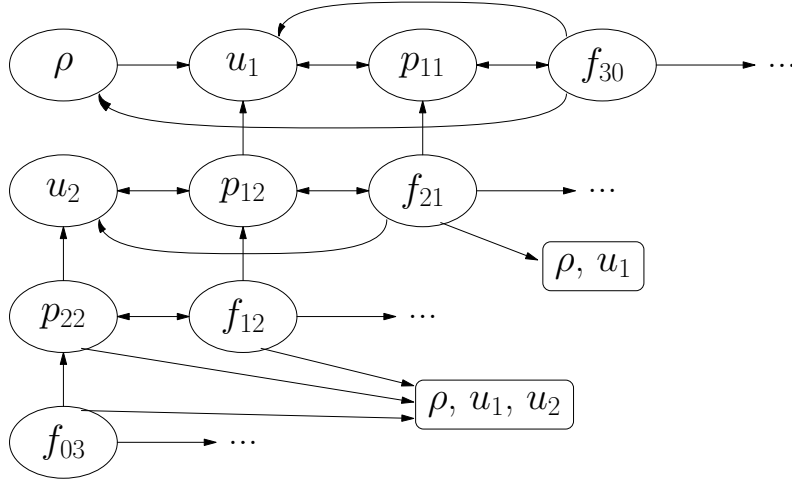


Figure 2: The dependency relationship of  $\mathbf{w}$  with  $D = 2$  and  $\frac{\partial \cdot}{\partial x_2} = 0$ .

It is interesting that in Fig. 2 there exists a path from every node to every other node in the same row along the direction of the arrow (e.g. there is a path between any two entries of  $\rho, u_1, p_{11}, f_{30}, \dots, f_{Me_1}$ ), while there is no path from one node to any other node in the next row (e.g. there is no path from  $\rho$  to  $u_2$ ). This indicates that the matrix  $\mathbf{A}_M$  is reducible (see Page. 288-289 of [10] for details). Thus by Fig. 2, if we rearrange  $\mathbf{w}$  by the lexicographic order of  $(\alpha_2, \alpha_1)$ , i.e.

$$\mathbf{w}' = (\underbrace{\rho, u_1, p_{11}/2, f_{3e_1}, \dots, f_{Me_1}}_{\text{first row}}, \underbrace{u_2, p_{12}, \dots, f_{(M-1)e_1+e_2}}_{\text{second row}}, \dots, \underbrace{f_{Me_2}}_{\text{last row}})^T,$$

the coefficient matrix  $\mathbf{A}_M$  can be collected into a block lower triangular matrix. Fig. 3(b) shows the permutation of  $\mathbf{w}'$  with  $D = 2$  and  $M = 8$ .

The permutation above shows that there exists a permutation matrix  $\mathbf{P}$  such that  $\mathbf{w}' = \mathbf{P}\mathbf{w}$ . Let  $\mathbf{A}'_M = \mathbf{P}\mathbf{A}_M\mathbf{P}^{-1}$ , then

$$\frac{D\mathbf{w}'}{Dt} + \mathbf{A}'_M \frac{\partial \mathbf{w}'}{\partial x_1} = \nu \mathbf{P} \mathbf{Q} \mathbf{P}^{-1} \mathbf{w}'$$

holds. Fig. 4 gives the sparsity pattern of  $\mathbf{A}'_M$  with  $M = 8$ . By Fig. 4, it is clear that  $\mathbf{A}'_M$  is reducible, and furthermore it is a block lower triangular matrix. Precisely,

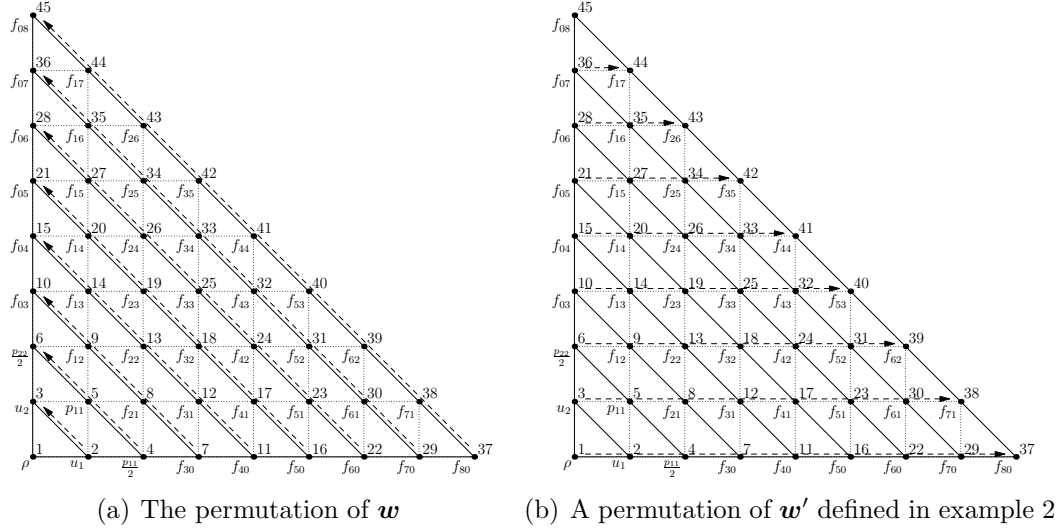


Figure 3: The permutation of the coefficients while  $D = 2, M = 8$ . Each node stands for one coefficient. The marks in the lower-left of the node shows the expression of the coefficient, while the number in the upper-right represents the ordinal number in  $\mathbf{w}$  or  $\mathbf{w}'$ . The dashed arrows depict the path of the corresponding permutation. The left one is the permutation of  $\mathbf{w}$ , and the right one is a permutation of  $\mathbf{w}'$  defined in example 2.

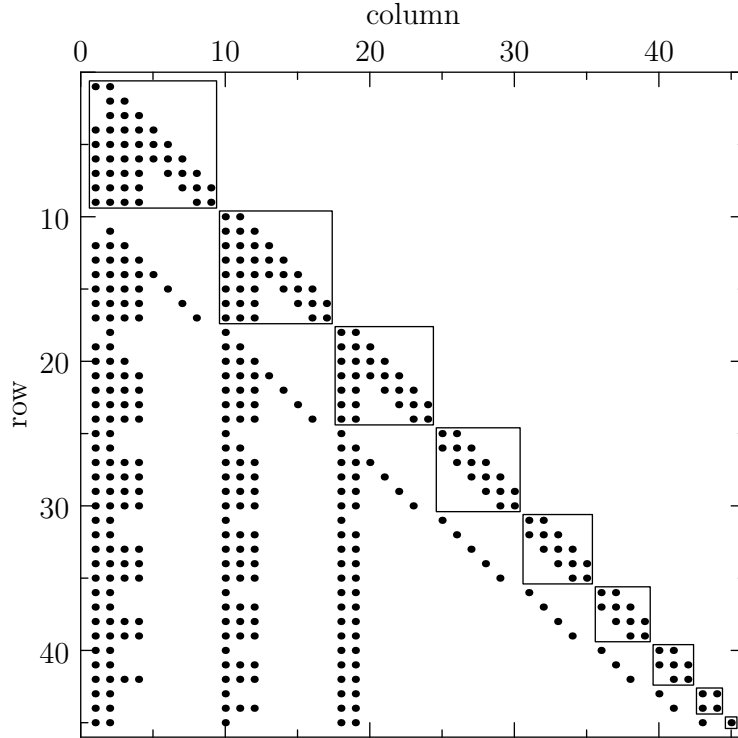


Figure 4: The sparsity pattern of  $\mathbf{A}'_M$  with  $M = 8, D = 2$ .  $\mathbf{A}'_M$  is reducible and a block lower triangular matrix. Each diagonal block is a lower Hessenberg matrix.

$\mathbf{A}'_M$  can be written as

$$\mathbf{A}'_M = \begin{bmatrix} \hat{A}_0 & & & \\ * & \hat{A}_1 & & \\ * & * & \hat{A}_2 & \\ \dots & \dots & \dots & \dots \\ * & * & * & \hat{A}_M \end{bmatrix},$$

where  $\hat{A}_i \in \mathbb{R}^{(M+1-i) \times (M+1-i)}$ ,  $i = 0, \dots, M$ , is a lower Hessenberg matrix.

Let us turn to study the properties of  $\hat{A}_i$ ,  $i = 0, \dots, M$ .  $\hat{A}_0$ ,  $\hat{A}_1$  and  $\hat{A}_2$  are defined in (3.12), (3.13) and (3.14), respectively.

$$\hat{A}_0 = \left( \begin{array}{cccccccccccccccc} U & \rho & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & U & 2\rho^{-1} & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 3p_{11}/2 & U & 3 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ -1/2\theta_{11}^2 & 4f_{3e_1} & \theta_{11} & U & 4 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ -\frac{5\theta_{11}f_{3e_1}}{2\rho} & 5f_{4e_1} & 3\frac{f_{3e_1}}{\rho} & \theta_{11} & U & 5 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ -3\frac{\theta_{11}f_{4e_1}}{\rho} & 6f_{5e_1} & 4\frac{f_{4e_1}}{\rho} & \frac{-3f_{3e_1}}{\rho} & \theta_{11} & U & 6 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{M\theta_{11}f_{(M-2)e_1}+\theta_{11}^2f_{(M-4)e_1}}{2\rho} & Mf_{(M-1)e_1} & \frac{(M-2)f_{(M-2)e_1}+\theta_{11}f_{(M-4)e_1}}{\rho} & \frac{-3f_{(M-3)e_1}}{\rho} & 0 & \dots & 0 & \theta_{11} & U & M & \dots & \dots & \dots & \dots \\ -\frac{(M+1)\theta_{11}f_{(M-1)e_1}+\theta_{11}^2f_{(M-3)e_1}}{2\rho} & (M+1)f_{Me_1} & \frac{(M-1)f_{(M-1)e_1}+\theta_{11}f_{(M-3)e_1}}{\rho} & \frac{-3f_{(M-2)e_1}}{\rho} & 0 & \dots & \dots & 0 & \theta_{11} & U & \dots & \dots & \dots & \dots \end{array} \right), \quad (3.12)$$

$$\hat{A}_1 = \left( \begin{array}{cccccccccccccccc} U & \rho^{-1} & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ p_{11} & U & 2 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 3f_{3e_1} & \theta_{11} & U & 3 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 4f_{4e_1} & \frac{3f_{3e_1}}{\rho} & \theta_{11} & U & 4 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 5f_{5e_1} & \frac{4f_{4e_1}}{\rho} & -\frac{2f_{3e_1}}{\rho} & \theta_{11} & U & 5 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (M-1)f_{(M-1)e_1} & \frac{(M-2)f_{(M-2)e_1}+\theta_{11}f_{(M-4)e_1}}{\rho} & -\frac{2f_{(M-3)e_1}}{\rho} & 0 & \dots & 0 & \theta_{11} & U & M-1 & \dots & \dots & \dots & \dots & \dots \\ Mf_{Me_1} & \frac{(M-1)f_{(M-1)e_1}+\theta_{11}f_{(M-3)e_1}}{\rho} & -\frac{2f_{(M-2)e_1}}{\rho} & 0 & \dots & \dots & 0 & \theta_{11} & U & \dots & \dots & \dots & \dots & \dots \end{array} \right), \quad (3.13)$$



$$\hat{A}_2 = \begin{pmatrix} U & 1 & 0 & \dots & 0 \\ \theta_{11} & U & 2 & 0 & \dots & 0 \\ \frac{3f_{3e_1}}{\rho} & \theta_{11} & U & 3 & 0 & \dots & 0 \\ \frac{4f_{4e_1}}{\rho} & -\frac{f_{3e_1}}{\rho} & \theta_{11} & U & 4 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{(M-2)f_{(M-2)e_1} + \theta_{11}f_{(M-4)e_1}}{\rho} & -\frac{f_{(M-3)e_1}}{\rho} & 0 & \dots & 0 & \theta_{11} & U & M-2 \\ \frac{(M-1)f_{(M-1)e_1} + \theta_{11}f_{(M-3)e_1}}{\rho} & -\frac{f_{(M-2)e_1}}{\rho} & 0 & \dots & 0 & \theta_{11} & U & \end{pmatrix}, \quad (3.14)$$

and for  $\hat{A}_i$ ,  $i = 3, \dots, M$ , they all have exactly the same form as

$$\hat{A}_i = \begin{pmatrix} U & 1 & 0 & \dots & 0 \\ \theta_{11} & U & 2 & 0 & \dots & 0 \\ 0 & \theta_{11} & U & 3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \theta_{11} & U & M-i \\ 0 & \dots & 0 & \theta_{11} & U & \end{pmatrix}, \quad i = 3, \dots, M, \quad (3.15)$$

where  $U = 0$ . Consider the matrix  $\mathbf{B}_M$  in (3.8), and denote  $\mathbf{B}_M(\rho, \theta_{11}, f_3, \dots, f_M) = \mathbf{B}_M$ , then

$$\hat{A}_0 = \mathbf{B}_M(\rho, \theta_{11}, f_{3e_1}, \dots, f_{Me_1}).$$

Hence,  $\hat{A}_0$  has exactly the same structure as  $\mathbf{B}_M$ .

**Example 3.** The properties of  $\hat{A}_0$  is listed in Section 3.1.1. Now let us study the properties of  $\hat{A}_1$  and  $\hat{A}_2$ .

It is clear that the coefficient matrix  $\hat{A}_1$  and  $\hat{A}_2$

- are independent of  $u$  and the diagonal entries are all zeros;
- are lower Hessenberg matrices.

Then we study the characteristic polynomials of  $\hat{A}_1$  and  $\hat{A}_2$ . Let  $\mathbf{r} \neq 0$  be an eigenvector of  $\hat{A}_1$  corresponding to the eigenvalue  $\lambda$ , e.g.  $\hat{A}_1 \mathbf{r} = \lambda \mathbf{r}$ . Since  $\hat{A}_1$  is a lower Hessenberg matrix, we assert  $r_1 \neq 0$ . Assume  $r_1 = 1$ , then  $\hat{A}_1(1, :) \mathbf{r} = \lambda r_1$  gives  $r_2 = \rho \lambda$ . Using the same skill on  $\hat{A}_1(k, :) \mathbf{r} = \lambda r_k$ ,  $k = 2, \dots, M-1$ , we can obtain

$$r_{k+1} = \rho \frac{He_k^{[\theta]}(\lambda)}{k!} - f_{ke_1} - \lambda f_{(k-1)e_1}, \quad k = 2, \dots, M-1.$$

$\hat{A}_1(M, :) \mathbf{r} = \lambda r_M$  can be written as

$$\rho \frac{He_M^{[\theta]}(\lambda)}{M!} - (-1)^M (f_{Me_1} - \lambda f_{(M-1)e_1}) = 0.$$

Hence, the above equation has to be satisfied, if  $\lambda$  is an eigenvalue of  $\hat{A}_1$ , which indicates the characteristic polynomial of  $\hat{A}_1$  is

$$He_M^{[\theta]}(\lambda) - (-1)^M M! (f_{Me_1} - \lambda f_{(M-1)e_1}) / \rho. \quad (3.16)$$

Similarly, the characteristic polynomial of  $\hat{A}_2$  is

$$He_{M-1}^{[\theta]}(\lambda) + (-1)^M (M-1)! f_{(M-1)e_1} / \rho. \quad (3.17)$$

For  $\alpha \in \mathbb{N}^D$ , let

$$\hat{\alpha} = (\alpha_2, \dots, \alpha_D), \quad (3.18)$$

and we denote  $\hat{e}_1 = 0$ ,  $\hat{e}_2 = (1, 0, \dots, 0) \in \mathbb{R}^{D-1}$ ,  $\dots$ ,  $\hat{e}_D = (0, \dots, 0, 1) \in \mathbb{R}^{D-1}$ .

For any  $2 \leq M \in \mathbb{N}$ , and any  $D \in \mathbb{N}^+$ , let  $\mathbf{P} \in \mathbb{R}^{N \times N}$  be the permutation matrix that  $\mathbf{w}' = \mathbf{P}\mathbf{w}$  in the lexicographic order of  $(\alpha_2, \dots, \alpha_D, \alpha_1)$ , and  $\mathbf{A}'_M = \mathbf{P}\mathbf{A}_M\mathbf{P}$ . Then we have the following results.

**Property 3.** For  $D \geq 2$ , the matrix  $\mathbf{A}'_M$  is a block lower triangular matrix, and the diagonal blocks

$$(\mathbf{A}'_M)_{ii} = \hat{A}_{\hat{\alpha}}$$

are irreducible, where  $i = \mathcal{N}_{D-1}(\hat{\alpha})$ . Precisely,  $\hat{A}_{\hat{\alpha}} = \hat{A}_{|\hat{\alpha}|}$ ,  $|\hat{\alpha}| = \alpha_2 + \dots + \alpha_D$ , where  $\hat{A}_i$ ,  $i = 0, \dots, M$ , is defined in Example 2.

### 3.2 Lack of global hyperbolicity

We are ready to show the major result in this section, that the moment system obtained is not globally hyperbolic for any  $D \in \mathbb{N}^+$ ,  $M \geq 3$ .

**Theorem 3.1.** The moment system obtained in Section 2 is not globally hyperbolic for any  $D \in \mathbb{N}^+$  and  $M \geq 3$ .

*Proof.* To prove the theorem, we need only to prove  $\mathbf{A}_M^{(1)}$  is not always diagonalizable with real eigenvalues.

Since  $\mathbf{P}$  is a permutation matrix, it is enough to examine  $\mathbf{A}'_M$ . Property 3 shows  $\mathbf{A}'_M$  is a block lower triangular matrix, thus if  $(\mathbf{A}'_M)_{11}$  is not diagonalizable with real eigenvalues,  $\mathbf{A}'_M$  is also not. Since  $(\mathbf{A}'_M)_{11} = \mathbf{B}_M(\rho, \theta_{11}, f_{3e_1}, \dots, f_{Me_1})$  and (3.9) indicates that if  $f_{Me_1}$  and  $f_{(M-1)e_1}$  take certain values,  $(\mathbf{A}'_M)_{11}$  has complex eigenvalues. This proves the theorem.  $\square$

For the case  $M = 2$ , if  $D = 1$ , then the moment system obtained is exactly the Euler equations, which is hyperbolic. If  $D = 3$ , then the moment system is the well-known 10-moment system, which has been studied in, e.g. [3, 25, 13].

## 4 Globally Hyperbolic Regularization

In this section, we propose a regularization to the moment system to obtain a globally hyperbolic moment system, following the idea in [4].

## 4.1 In one-dimensional spatial space

For any  $3 \leq M \in \mathbb{N}$ , the generalized Grad-type moment system obtained in Section. 2 gives *accurate* evolution equations for all the variables except for those  $f_\alpha$  with  $|\alpha| = M$ , since  $f_{\alpha+e_d}$ ,  $d = 1, \dots, D$  appear in the equations of them, and are taken to be zero in Grad's closure. The regularization methods given in such as [18, 24, 28] were trying to propose a modified form for  $f_{\alpha+e_d}$ ,  $|\alpha| = M$ . Actually, noticing that the terms  $f_{\alpha+e_d}$ ,  $|\alpha| = M$ , appear only in the evolving equation of  $f_\alpha$ ,  $|\alpha| = M$  in the form of its derivatives, a reasonable regularization should only modify the evolving equations of  $f_\alpha$ ,  $|\alpha| = M$  by proposing a suitable form of the derivatives  $\partial f_{\alpha+e_d}/\partial x_d$ ,  $|\alpha| = M$ ,  $d = 1, \dots, D$ . Property 3 show us that the coefficient matrix  $\mathbf{A}'_M$  has the form

$$\mathbf{A}'_M = \begin{pmatrix} \hat{A}_0 & 0 \\ * & * \end{pmatrix}, \quad (4.1)$$

since the variables  $u_1$ ,  $\theta_{11}$  and  $f_{ke_i}$ ,  $k = 0, \dots, M$  are independent of the other variables. It is natural to require the regularization to preserve such structure. The regularization we are proposing below can fulfil all these constraints, and at the same time achieves the global hyperbolicity. For convenience, we call

**Definition 4.1.** *A regularization for the generalized Grad-type moment system is admissible, if*

1. *it only modifies the governing equations of  $f_\alpha$ ,  $|\alpha| = M$ ;*
2. *it keeps the regularized coefficient matrix have the form as (4.1).*

The proof of Theorem 3.1 shows that  $(\mathbf{A}'_M)_{ii}$ ,  $i = 1, \dots, \hat{N}$  is diagonalizable with real eigenvalues is a necessary condition for that  $\mathbf{A}_M$  is diagonalizable with real eigenvalues. In this subsection, we first study the regularization of  $(\mathbf{A}'_M)_{ii}$ ,  $i = 1, \dots, \hat{N}$ , then prove that the regularization also make  $\mathbf{A}_M$  diagonalizable with real eigenvalues.

As discussed above, only the last row of  $\hat{A}_0$  are to be modified in the regularization. Property 3 shows  $\hat{A}_0 = \mathbf{B}_M(\rho, \theta_{11}, f_{3e_1}, \dots, f_{Me_1})$ . And for  $D = 1$ , the coefficient matrix  $\mathbf{A}'_M = \hat{A}_0$ . In [5], the regularization with  $D = 1$  is studied in details, and the result therein we will need later on is as below.

**Lemma 4.2.** *Let*

$$\tilde{\mathbf{B}}_M \frac{\partial \mathbf{w}}{\partial x} = \mathbf{B}_M \frac{\partial \mathbf{w}}{\partial x} - (M+1) \left( f_{Me_1} \frac{\partial u}{\partial x} + \frac{f_{M-1}}{2\rho} \left( \frac{\partial p}{\partial x} - \theta \frac{\partial \rho}{\partial x} \right) \right) I_{M+1},$$

*for any admissible  $\mathbf{w}$ , i.e.,*

$$\tilde{\mathbf{B}}_M = \mathbf{B}_M - I_{M+1} \mathcal{R}_0^T, \quad (4.2)$$

*where  $\mathcal{R}_0 = (M+1)(-\theta f_{M-1}/2\rho, f_M, f_{M-1}/\rho, 0, \dots, 0)^T \in \mathbb{R}^{M+1}$  and  $I_{M+1}$  is the last column of the  $(M+1) \times (M+1)$  identity matrix. Then  $\tilde{\mathbf{B}}_M$  is diagonalizable with real eigenvalues. Precisely, the characteristic polynomial of  $\tilde{\mathbf{B}}_M$  is*

$$\det(\lambda \mathbf{I} - \tilde{\mathbf{B}}_M) = \theta^{M+1} He_{M+1}^{[\theta]}(\lambda),$$

and the eigenvalues of  $\tilde{\mathbf{B}}_M$  are  $\sqrt{\theta}C_{1,M+1}, \dots, \sqrt{\theta}C_{M+1,M+1}$ , where  $C_{j,k}$  is the  $j$ -th root of Hermite polynomial  $He_k(x)$ , noticing that  $He_k(x)$ ,  $k \in \mathbb{N}$  has  $k$  different zeros, which read  $C_{1,k}, \dots, C_{k,k}$ , and satisfy  $C_{1,k} < \dots < C_{k,k}$ . Let  $\mathbf{r} \in \mathbb{R}^{M+1}$  and

$$\begin{aligned} r_1 &= 1, \quad r_2 = \lambda/\rho, \quad r_3 = \lambda^2/2, \\ r_k &= He_{k-1}^{[\theta]}(\lambda)/(k-1)! - \lambda f_{k-2}/\rho - (\lambda^2 - 1)f_{k-3}/(2\rho), \quad k = 4, \dots, M+1, \end{aligned}$$

where  $\lambda$  is an eigenvalue of  $\mathbf{B}_M$ , then  $\mathbf{r}$  is an eigenvector of  $\mathbf{B}_M$  for the eigenvalue  $\lambda$ .

Moreover, the regularization is admissible and the admissible regularization to modify  $\mathbf{B}_M$  to be diagonalizable with real eigenvalues with the characteristic polynomial  $\theta^{M+1}He_{M+1}^{[\theta]}$  is unique.

*Remark 1.* Since  $f_\alpha$  are related to  $f(t, \mathbf{x}, \boldsymbol{\xi})$  by (2.9), the positivity of the distribution function will impose some constraints on the  $f_\alpha$ . Particularly,  $\rho$  and  $\Theta$  satisfy

$$\rho > 0 \quad \text{and} \quad \Theta \text{ being a symmetrical positive definite matrix.} \quad (4.3)$$

Though (4.3) is not enough to ensure the positivity of  $f(t, \mathbf{x}, \boldsymbol{\xi})$ , the discussion in this section requires no further constraints on all the other variables. Hence, the admissible  $\mathbf{w}'$  stands for the  $\mathbf{w}'$  satisfying (4.3) in this section.

We extend the results of  $D = 1$  to any dimensional case.

**Definition 4.3.**  $\tilde{\mathbf{A}}_M$  is called the regularized matrix of  $\mathbf{A}_M$ , if it satisfies that for any admissible  $\mathbf{w}$ ,

$$\begin{aligned} \tilde{\mathbf{A}}_M \frac{\partial \mathbf{w}}{\partial x_1} &= \mathbf{A}_M \frac{\partial \mathbf{w}}{\partial x_1} \\ &- \sum_{|\alpha|=M} (\alpha_1 + 1) \left( \sum_{i=1}^D f_{\alpha+e_1-e_i} \frac{\partial u_i}{\partial x_1} + \sum_{i,j=1}^D \frac{f_{\alpha+e_1-e_i-e_j}}{2\rho} \left( \frac{\partial p_{ij}}{\partial x_1} - \theta_{ij} \frac{\partial \rho}{\partial x_1} \right) \right) I_{\mathcal{N}_D(\alpha)}, \end{aligned} \quad (4.4)$$

where  $I_k$  is the  $k$ -th column of the  $N \times N$  identity matrix.

In this the definition of the regularized matrix  $\tilde{\mathbf{A}}_M$ ,  $\tilde{\mathbf{A}}_M$  is obtained by changing a few entries of  $\mathbf{A}_M$ . Precisely for any  $|\alpha| = M$ , let  $k = \mathcal{N}_D(\alpha)$

$$\begin{aligned} \tilde{\mathbf{A}}_M(k, 1) &= \mathbf{A}_M(k, 1) + (\alpha_1 + 1) \sum_{i,j=1}^D \frac{\theta_{ij} f_{\alpha+e_1-e_i-e_j}}{2\rho}, \\ \tilde{\mathbf{A}}_M(k, d+1) &= \mathbf{A}_M(k, d+1) - (\alpha_1 + 1) f_{\alpha+e_1-e_d}, \quad d = 1, \dots, D, \\ \tilde{\mathbf{A}}_M(k, \mathcal{N}_D(e_i + e_j)) &= \mathbf{A}_M(k, \mathcal{N}_D(e_i + e_j)) - (\alpha_1 + 1) \frac{f_{\alpha+e_1-e_i-e_j}}{\rho} \quad i, j = 1, \dots, D. \end{aligned}$$

Other entries of  $\tilde{\mathbf{A}}_M$  remain the same values as those of  $\mathbf{A}_M$ .

For convenience, we list the regularized collisionless moment system with  $\frac{\partial \cdot}{\partial x_2} = \dots = \frac{\partial \cdot}{\partial x_D} = 0$ , which is the case of 1D spatial space, as following:

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_1}{\partial x_1} = 0, \quad (4.5a)$$

$$\frac{Du_i}{Dt} + \frac{1}{\rho} \frac{\partial p_{1i}}{\partial x_1} = 0, \quad (4.5b)$$

$$\frac{Dp_{ij}}{Dt} + p_{ij} \frac{\partial u_1}{\partial x_1} + p_{1i} \frac{\partial u_j}{\partial x_1} + p_{1j} \frac{\partial u_i}{\partial x_1} + (e_i + e_j + e_1)! \frac{\partial f_{e_i+e_j+e_1}}{\partial x_1} = 0, \quad (4.5c)$$

$$\begin{aligned} \frac{Df_\alpha}{Dt} + \sum_{k=1}^D \theta_{1k} \frac{\partial f_{\alpha-e_k}}{\partial x_1} + (1 - \delta_{|\alpha|,M})(\alpha_1 + 1) \frac{\partial f_{\alpha+e_1}}{\partial x_1} \\ + \sum_{i,j=1}^D \frac{\tilde{C}_{ij}(\alpha)}{2\rho} \left( \frac{\partial p_{ij}}{\partial x_1} - \theta_{ij} \frac{\partial \rho}{\partial x_1} \right) + \sum_{i=1}^D (1 - \delta_{|\alpha|,M})(\alpha_1 + 1) f_{\alpha-e_i+e_1} \frac{\partial u_i}{\partial x_1} \\ - \sum_{i=1}^D \frac{f_{\alpha-e_i}}{\rho} \frac{\partial p_{i1}}{\partial x_1} - \sum_{i,j=1}^D \frac{(e_i + e_j + e_1)!}{2} \frac{f_{\alpha-e_i-e_j}}{\rho} \frac{\partial f_{e_i+e_j+e_1}}{\partial x_1} = 0, \end{aligned} \quad (4.5d)$$

where  $\tilde{C}_{ij}$  is

$$\tilde{C}_{ij}(\alpha) = \sum_{k=1}^D \theta_{k1} f_{\alpha-e_i-e_j-e_k} + (1 - \delta_{|\alpha|,M})(\alpha_1 + 1) f_{\alpha-e_i-e_j+e_1}. \quad (4.6)$$

Clearly, the equations (4.5a), (4.5b), (4.5c) and (4.5d) is the simplified formulation of the regularized moment system, and the entries of the matrix  $\tilde{\mathbf{A}}'_M$  can be retrieved directly from the system.

Notice that  $\tilde{\mathbf{A}}'_M = \mathbf{P} \tilde{\mathbf{A}}_M \mathbf{P}^{-1}$  is the regularized matrix of  $\mathbf{A}'_M$ , where  $\mathbf{P}$  is the permutation matrix, such that  $\mathbf{A}'_M = \mathbf{P} \mathbf{A}_M \mathbf{P}^{-1}$ . Clearly, only the rows of  $\tilde{\mathbf{A}}'_M$  corresponding to the last rows of  $\hat{A}_k$ ,  $k = 1, \dots, \hat{N}$  are different from those of  $\mathbf{A}'_M$ . Particularly,  $\tilde{\mathbf{B}}_M$  is defined in (4.2), and  $\tilde{\hat{A}}_1$  and  $\tilde{\hat{A}}_2$  are denoted by

$$\tilde{\hat{A}}_1 = \hat{A}_1 - I_M^{(M)} \mathcal{R}_1^T, \quad (4.7a)$$

$$\tilde{\hat{A}}_2 = \hat{A}_2 - I_{M-1}^{(M-1)} \mathcal{R}_2^T, \quad (4.7b)$$

where  $I_k^{(n)}$  is the  $k$ -th column of the  $n \times n$  identity matrix, and

$$\begin{aligned} \mathcal{R}_1 &= M(f_{Me_1}, f_{(M-1)e_1}/\rho, 0, \dots, 0)^T \in \mathbb{R}^M, \\ \mathcal{R}_2 &= (M-1)(f_{(M-1)e_1}/\rho, 0, \dots, 0)^T \in \mathbb{R}^{M-1}. \end{aligned}$$

For  $\hat{A}_k$ ,  $k = 3, \dots, M$ , we have  $\tilde{\hat{A}}_k = \hat{A}_k$ . Hence, the regularization (4.4) is admissible.

Here we give a note on the convention of the notations used here.  $\mathbf{A}_M$  is the coefficient of the moment system (3.7) on the direction  $x$ , and  $\mathbf{A}'_M = \mathbf{P}\mathbf{A}_M\mathbf{P}^{-1}$  is a lower block triangular matrix, where  $\mathbf{P}$  satisfies  $\mathbf{w}' = \mathbf{P}\mathbf{w}$ .  $\hat{\alpha} = (\alpha_2, \dots, \alpha_D)$  and  $\hat{A}_{\hat{\alpha}}$ ,  $|\hat{\alpha}| \leq M$  are diagonal blocks of  $\mathbf{A}'_M$  defined in Property 3.  $\tilde{\cdot}$  stands for the regularized matrix, such as,  $\tilde{\mathbf{A}}'_M$  is the regularized matrix of  $\mathbf{A}'_M$ , and  $\tilde{A}_{\hat{\alpha}}$  is the regularized matrix of  $\hat{A}_{\hat{\alpha}}$ .

Lemma (4.2) shows for  $D = 1$ , the regularization defined in Definition 4.3 make the coefficient matrix diagonalizable with real eigenvalues. For arbitrary dimensional case, we have the following results.

**Theorem 4.4.** *The regularized moment system*

$$\frac{D\mathbf{w}}{Dt} + \tilde{\mathbf{A}}_M \frac{\partial \mathbf{w}}{\partial x_1} = 0$$

is globally hyperbolic for any admissible  $\mathbf{w}$ .

To prove this theorem, we need to verify the regularized matrix  $\tilde{\mathbf{A}}_M$  is diagonalizable with real eigenvalues for any admissible  $\mathbf{w}$ . Since  $\tilde{\mathbf{A}}'_M$  is a similar matrix of  $\tilde{\mathbf{A}}_M$ , next we first study the eigenvalues and eigenvectors of  $\tilde{A}_k$ ,  $k = 1, \dots, M$ , then we can obtain the characteristic polynomial of  $\tilde{\mathbf{A}}'_M$ , and verify that all the eigenvalues of  $\tilde{\mathbf{A}}'_M$  are real. Furthermore, any eigenvector of  $\tilde{A}_k$ ,  $k = 0, \dots, M$  can be extended to an eigenvector of  $\tilde{\mathbf{A}}'_M$  under relevant constraints, and then we can prove  $\tilde{\mathbf{A}}'_M$  have  $N$ -linearly independent eigenvectors, which means  $\tilde{\mathbf{A}}'_M$  is diagonalizable.

Since  $\tilde{A}_1$  is a lower Hessenberg matrix, it is possible to calculate its eigenvector, once the eigenvalue is given. Actually, we have the following lemma.

**Lemma 4.5.** *The matrix  $\tilde{A}_1 \in \mathbb{R}^{M \times M}$  is diagonalizable with real eigenvalues for any  $\rho > 0$ ,  $\theta_{11} > 0$ ,  $f_{ke_1} \in \mathbb{R}$ ,  $k = 3, \dots, M-1$ . Precisely, its characteristic polynomial is*

$$\det(\lambda \mathbf{I} - \tilde{A}_1) = \theta_{11}^M He_M^{[\theta_{11}]}(\lambda), \quad (4.8)$$

and the eigenvalues of  $\tilde{A}_1$  are  $\sqrt{\theta_{11}}C_{1,M}, \dots, \sqrt{\theta_{11}}C_{M,M}$ . Let  $\mathbf{r} \in \mathbb{R}^M$  and

$$r_1 = 1, \quad r_2 = \rho\lambda, \quad r_k = \rho He_{k-1}^{[\theta_{11}]}(\lambda)/(k-1)! - f_{(k-1)e_1} - \lambda f_{(k-2)e_1}, \quad k = 2, \dots, M,$$

then  $\mathbf{r}$  is an eigenvector of  $\tilde{A}_1$  for the eigenvalue  $\lambda$ .

The proof is trivial but rather tedious, which is presented in the Appendix B. Analogously, the matrix  $\tilde{A}_2 \in \mathbb{R}^{(M-1) \times (M-1)}$  has the following properties.

**Lemma 4.6.** *The matrix  $\tilde{A}_2 \in \mathbb{R}^{(M-1) \times (M-1)}$  is diagonalizable with real eigenvalues for any  $\rho > 0$ ,  $\theta_{11} > 0$ ,  $f_{ke_1} \in \mathbb{R}$ ,  $k = 3, \dots, M-2$ . Precisely, its characteristic polynomial is*

$$\det(\lambda \mathbf{I} - \tilde{A}_2) = \theta_{11}^{M-1} He_{M-1}^{[\theta_{11}]}(\lambda), \quad (4.9)$$

and the eigenvalues of  $\tilde{A}_2$  are  $\sqrt{\theta_{11}}C_{1,M-1}, \dots, \sqrt{\theta_{11}}C_{M-1,M-1}$ . Let  $\mathbf{r} \in \mathbb{R}^{M-1}$  and

$$r_1 = 1, \quad r_k = He_{k-1}^{[\theta_{11}]}(\lambda)/(k-1)! - f_{(k-1)e_1}, \quad k = 2, \dots, M-1,$$

then  $\mathbf{r}$  is an eigenvector of  $\tilde{A}_2$  for the eigenvalue  $\lambda$ .

For the matrix  $\tilde{A}_n$ ,  $n = 3, \dots, M$ , we have the following results.

**Lemma 4.7.** *The matrix  $\tilde{A}_n$ ,  $n = 3, \dots, M$  is diagonalizable with real eigenvalues for any  $\theta_{11} > 0$ . Precisely, let  $m = M + 1 - n$ , then the characteristic polynomial of  $\tilde{A}_n$  is*

$$\det(\lambda \mathbf{I} - \tilde{A}_n) = \theta_{11}^m He_m^{[\theta_{11}]}(\lambda), \quad (4.10)$$

and the eigenvalues of  $\tilde{A}_n$  are  $\sqrt{\theta_{11}}C_{1,m}, \dots, \sqrt{\theta_{11}}C_{m,m}$ . Let  $\mathbf{r} \in \mathbb{R}^m$  and

$$r_k = He_{k-1}^{[\theta_{11}]}(\lambda)/(k-1)!, \quad k = 1, \dots, m,$$

then  $\mathbf{r}$  is an eigenvector of  $\tilde{A}_n$  for the eigenvalue  $\lambda$ .

The proof of these two lemmas are almost the same as that of Lemma 4.5 thus we omit it.

Property 3 and Lemma 4.2, 4.5, 4.6 and 4.7 show the regularized matrix  $\tilde{\mathbf{A}}'_M$  is a block lower triangular matrix and the characteristic polynomial of each diagonal block is known. Since  $\tilde{\mathbf{A}}'_M$  is a similar matrix of  $\tilde{\mathbf{A}}_M$ , we have the following result on the characteristic polynomial of  $\tilde{\mathbf{A}}_M$ .

**Lemma 4.8.** *Let*

$$\mathcal{P}_{1,m} = \theta_{11}^{m+1} He_{m+1}^{[\theta_{11}]}(\lambda), \quad m \in \mathbb{N}, \quad (4.11)$$

$$\mathcal{P}_{d,m} = \prod_{k=0}^m \mathcal{P}_{d-1,k}, \quad 1 < d \in \mathbb{N}^+. \quad (4.12)$$

$\mathcal{P}_{D,M}$  is the characteristic polynomial of  $\tilde{\mathbf{A}}_M$ .

*Proof.* If  $D = 1$ , it is part of Lemma 4.2. Next we consider the case  $D \geq 2$ . Since  $\tilde{\mathbf{A}}_M$  is similar to  $\tilde{\mathbf{A}}'_M$ , the characteristic polynomial of  $\tilde{\mathbf{A}}_M$  is same as that of  $\tilde{\mathbf{A}}'_M$ . Thus,

$$\begin{aligned} \det(\lambda \mathbf{I} - \tilde{\mathbf{A}}_M) &= \det(\lambda \mathbf{I} - \tilde{\mathbf{A}}'_M) \\ &= \prod_{|\hat{\alpha}| \leq M} \det(\lambda \mathbf{I} - \tilde{A}_{\hat{\alpha}}) \\ &= \prod_{m=0}^M \left( \theta_{11}^{m+1} He_{m+1}^{[\theta_{11}]} \right)^{\binom{D-2}{M-m+D-2}} \\ &= \mathcal{P}_{D,M}. \end{aligned}$$

The second equality is then obtained by induction on  $D$ . □

Until now, we have revealed that the regularized matrix  $\tilde{\mathbf{A}}'_M$  is a block lower triangular matrix and each diagonal block is diagonalizable with real eigenvalues. Unfortunately, it is not sufficient to conclude that  $\tilde{\mathbf{A}}'_M$  is diagonalizable yet, since some eigenvalues may be not semi-simple. This pushes us to clarify the structure of the eigen-subspace of  $\tilde{\mathbf{A}}'_M$ . Actually, we will demonstrate that based on any eigenvector of  $\tilde{\mathbf{A}}_k$ ,  $k = 0, \dots, M$ , we can construct an eigenvector of  $\tilde{\mathbf{A}}'_M$ . For this purpose, we start with an example.

**Example 4.** Consider the block lower triangular matrix

$$\mathbf{A} = \left( \begin{array}{ccc|cc} 1 & 14 & 36 & 0 & 0 \\ 16 & -10 & -54 & 0 & 0 \\ -10 & 4 & 27 & 0 & 0 \\ \hline 2 & 1 & 1 & 6 & 3 \\ 4 & 2 & 2 & 6 & 9 \end{array} \right) = \begin{pmatrix} A_{11} & \mathbf{0} \\ A_{21} & A_{22} \end{pmatrix}. \quad (4.13)$$

The eigenvalues and eigenvectors of  $A_{11}$  and  $A_{22}$  that

$$\begin{aligned} A_{11} : \quad & \lambda_1 = 3, \mathbf{r}_1 = (16, -26, 11)^T; \quad \lambda_2 = 6, \mathbf{r}_2 = (10, -17, 8)^T; \quad \lambda_3 = 9, \mathbf{r}_3 = (1, -2, 1)^T; \\ A_{22} : \quad & \lambda_4 = 3, \mathbf{r}_4 = (1, -1)^T; \quad \lambda_5 = 12, \mathbf{r}_5 = (1, 2)^T. \end{aligned}$$

Now we examine whether there is  $\mathbf{R}_i \in \mathbb{R}^5$  and  $\mathbf{R}_i(1:3) = \mathbf{r}_i$ ,  $i = 1, 2, 3$ , satisfying  $\mathbf{A}\mathbf{R}_i = \lambda_i \mathbf{R}_i$ . Actually, it is equivalent to whether there is a solution of  $A_{21}\mathbf{r}_i + (A_{22} - \lambda_i \mathbf{I})\mathbf{R}_i(4:5) = 0$ , which has a solution if and only if the augmented matrix of  $A_{22} - \lambda_i \mathbf{I}$  has the same rank as  $A_{22} - \lambda_i \mathbf{I}$ , i.e.

$$\text{rank}([A_{22} - \lambda_i \mathbf{I}, A_{21}\mathbf{r}_i]) = \text{rank}(A_{22} - \lambda_i \mathbf{I}). \quad (4.14)$$

For  $i = 2, 3$ , since  $A_{22} - \lambda_i \mathbf{I}$  is nonsingular, (4.14) holds. For  $i = 1$ , some simple calculations give that  $\text{rank}([A_{22} - \lambda_i \mathbf{I}, A_{21}\mathbf{r}_i]) = \text{rank}(A_{22} - \lambda_i \mathbf{I}) = 1$ .

Next we check whether there is  $\mathbf{R}_i \in \mathbb{R}^5$  and  $\mathbf{R}_i(4:5) = \mathbf{r}_i$ ,  $i = 4, 5$ , satisfying  $\mathbf{A}\mathbf{R}_i = \lambda_i \mathbf{R}_i$ . Actually,  $\mathbf{R}_i(1:3) = (a, b, -2a - b)$  satisfies the condition. Particularly, if  $a = b = 0$ ,  $\mathbf{R}_i(1:3) = 0$ .

Here we call  $\mathbf{R}_i$  is a prolongation of  $\mathbf{r}_i$ , and call the  $\mathbf{R}_i$  with  $\mathbf{R}_i(1:3) = 0$ ,  $i = 4, 5$  a proper prolongation of  $\mathbf{r}_i$ . It is obvious that  $\mathbf{R}_i$ ,  $i = 1, \dots, 5$ , is linearly independent.

**Definition 4.9.** For a  $k \times k$  block lower triangular matrix  $\mathbf{A} \in \mathbb{R}^N$  with the size of diagonal block  $n_i \times n_i$ ,  $n_1 + \dots + n_k = N$ ,  $\mathbf{r}_i$  is an eigenvector of the  $i$ -th diagonal block for the eigenvalue  $\lambda$ . We call  $\mathbf{R}$  is a prolongation of  $\mathbf{r}_i$ , if  $\mathbf{R}((n_1 + \dots + n_{i-1} + 1) : (n_1 + \dots + n_i)) = \mathbf{r}_i$  and  $\mathbf{A}\mathbf{R} = \lambda \mathbf{R}$ . Particularly,  $\mathbf{R}$  is a proper prolongation of  $\mathbf{r}_i$  if  $\mathbf{R}(1 : (n_1 + \dots + n_{i-1})) = 0$ .

**Property 4.**  $\mathbf{A}$  is defined same as that in Definition 4.9, and each diagonal block of  $\mathbf{A}$  is diagonalizable with real eigenvalue.  $\mathbf{r}_{i,1}, \dots, \mathbf{r}_{i,n_i}$  are eigenvectors of the  $i$ -th diagonal block of  $\mathbf{A}$ . If for each  $\mathbf{r}_{i,j}$ ,  $i = 1, \dots, k$ ,  $j = 1, \dots, n_i$ , there is a proper prolongation  $\mathbf{R}_{i,j}$ , then  $\mathbf{R}_{i,j}$  are linearly independent.



*Proof.* We permute  $\mathbf{R}_{i,j}$  by the order  $\vec{\mathbf{R}} = [\mathbf{R}_{1,1}, \dots, \mathbf{R}_{1,n_1}, \mathbf{R}_{2,1}, \dots, \mathbf{R}_{k,n_k}]$ . Since  $\mathbf{R}_{i,j}$ ,  $i = 1, \dots, k$ ,  $j = 1, \dots, n_i$  is a proper prolongation of  $\mathbf{r}_{i,j}$ ,  $\vec{\mathbf{R}}$  is a block lower triangular matrix. For a fixed  $i \in \{1, \dots, k\}$ ,  $\mathbf{r}_{i,j}$ ,  $j = 1, \dots, n_i$  are linearly independent. Hence each diagonal block of  $\vec{\mathbf{R}}$  are nonsingular, thus  $\vec{\mathbf{R}}$  is nonsingular, which indicates  $\mathbf{R}_{i,j}$ ,  $i = 1, \dots, k$ ,  $j = 1, \dots, n_i$  are linearly independent.  $\square$

Next we check whether there is a proper prolongation of each eigenvector of every diagonal block of  $\tilde{\mathbf{A}}'_M$  and get the following result.

**Lemma 4.10.**  $\mathbf{r}_\alpha$  is an eigenvector of  $\hat{A}_{\hat{\alpha}} \in \mathbb{R}^{(M+1-|\hat{\alpha}|) \times (M+1-|\hat{\alpha}|)}$ , for the  $\alpha_1$ -th eigenvalue  $\lambda = \sqrt{\theta_{11}} C_{\alpha_1, M+1-|\hat{\alpha}|}$ , then there is a proper prolongation  $\mathbf{R}_\alpha$  satisfying  $\tilde{\mathbf{A}}'_M \mathbf{R}_\alpha = \lambda \mathbf{R}_\alpha$ .

The proof of the lemma is rather long and tedious, so we move the proof in Appendix C.

Clearly, this lemma is essential to prove Theorem 4.4. With all these preparation, the proof of Theorem 4.4 as follows is straight forward:

*Proof of the Theorem 4.4.* Lemma 4.8 shows all the eigenvalues of  $\tilde{\mathbf{A}}_M$  are real. Since  $\tilde{\mathbf{A}}'_M$  is similar to  $\tilde{\mathbf{A}}_M$ , all the eigenvalues of  $\tilde{\mathbf{A}}'_M$  are also real. Lemma 4.2, 4.5, 4.6 and 4.7 show that each diagonal block of  $\tilde{\mathbf{A}}'_M$  is diagonalizable with real eigenvalues, and Lemma 4.10 indicates each eigenvector of each diagonal block can be extended to an eigenvector of  $\tilde{\mathbf{A}}'_M$  by a proper prolongation. Hence considering Property 4, we obtain that  $\tilde{\mathbf{A}}'_M$  is diagonalizable with real eigenvalues. This finishes the proof.  $\square$

In addition, for the eigenvector of  $\tilde{\mathbf{A}}'_M$ , we define  $\mathbf{R} = (R_\alpha)^T \in \mathbb{R}^N$ , where  $R_\alpha$  is permuted by the lexicographic order of  $(\alpha_2, \dots, \alpha_D, \alpha_1)$ , same as that of  $\mathbf{w}'$ . Particularly, the first entry of  $\mathbf{R}$  is  $R_0$ . Hence we have  $\mathbf{P}\mathbf{R} = (R_\alpha)^T \in \mathbb{R}^N$ , where  $\mathbf{P}$  satisfies  $\mathbf{w}' = \mathbf{P}\mathbf{w}$ , and  $R_\alpha$  is permuted same as that of  $\mathbf{w}$ .

Before we end this subsection, we give a corollary of Lemma 4.10, which will be used in Section 5.

**Corollary 4.11.** Let  $\mathbf{R} \neq 0$  be a right eigenvector of the matrix  $\tilde{\mathbf{A}}'_M$  for the eigenvalue  $\lambda$ . Then

$$\lambda R_0 \neq 0 \text{ holds if and only if } \hat{H}e_{M+1}^{[\theta_{11}]}(\lambda) = 0 \text{ and } \lambda \neq 0,$$

where  $\hat{H}e_{M+1}^{[\theta_{11}]}(\lambda)$  is the characteristic polynomial of  $\tilde{\hat{A}}_0$ .

*Proof.* Let  $\mathbf{r} = (R_0, R_{e_1}, \dots, R_{M_{e_1}})^T \in \mathbb{R}^{M+1}$ . Since  $\tilde{\mathbf{A}}'_M$  is a block lower triangular matrix, and the first diagonal block is  $\tilde{\hat{A}}_0$ , we have  $\tilde{\hat{A}}_0 \mathbf{r} = \lambda \mathbf{r}$ .

“ $\Rightarrow$ ” Since  $R_0 \neq 0$  and  $\lambda \neq 0$ ,  $\mathbf{r} \neq 0$  is an eigenvector of  $\tilde{\hat{A}}_0$  for the eigenvalue  $\lambda$ , which indicates  $\hat{H}e^{[\theta_{11}]}(\lambda) = 0$ . Thus we have  $\hat{H}e^{[\theta_{11}]}(\lambda) = 0$  and  $\lambda \neq 0$ .

“ $\Leftarrow$ ”  $\hat{H}e^{[\theta_{11}]}(\lambda) = 0$  and  $\lambda \neq 0$  mean  $\lambda$  is an eigenvalue of  $\tilde{\hat{A}}_0$ . Since each nonzero eigenvalue of  $\tilde{\hat{A}}_0$  is simple eigenvalue of  $\tilde{\mathbf{A}}'_M$ , we get  $\mathbf{r} \neq 0$ . (3.10) indicates  $R_0 \neq 0$ , so  $R_0 \lambda \neq 0$ .  $\square$

## 4.2 In multi-dimensional spatial space

In this subsection, we give the general hyperbolic moment system containing all moments with orders not more than  $M$ . Without the assumption that the distribution function  $f$  is independent on  $x_2, \dots, x_D$ , the collisionless moment system obtained in Section 2 can be written as

$$\frac{D\mathbf{w}}{Dt} + \sum_{d=1}^D \mathbf{A}_M^{(d)} \frac{\partial \mathbf{w}}{\partial x_d} = 0, \quad (4.15)$$

where  $\mathbf{w}$  and  $\mathbf{A}_M^{(d)}$ ,  $d = 1, \dots, D$  are same as that in (3.7). And particularly,  $\mathbf{A}_M^{(1)}$  is the matrix  $\mathbf{A}_M$  discussed in Section 3.1, 3.2 and 4.1. Similar as Definition 4.3, we give the following definition:

**Definition 4.12.** For  $d = 1, \dots, D$ ,  $\tilde{\mathbf{A}}_M^{(d)}$  is called the regularized matrix  $\mathbf{A}_M^{(d)}$ , if it satisfies that for any admissible  $\mathbf{w}$ ,

$$\begin{aligned} \tilde{\mathbf{A}}_M^{(d)} \frac{\partial \mathbf{w}}{\partial x_d} &= \mathbf{A}_M^{(d)} \frac{\partial \mathbf{w}}{\partial x_d} \\ &\quad - \sum_{|\alpha|=M} (\alpha_d + 1) \left( \sum_{i=1}^D f_{\alpha+e_d-e_i} \frac{\partial u_i}{\partial x_d} + \sum_{i,j=1}^D \frac{f_{\alpha+e_d-e_i-e_j}}{2\rho} \left( \frac{\partial p_{ij}}{\partial x_d} - \theta_{ij} \frac{\partial \rho}{\partial x_d} \right) \right) I_{N_D(\alpha)}, \end{aligned} \quad (4.16)$$

where  $I_k$  is the  $k$ -th column of the  $N \times N$  identity matrix.

Then the multi-dimensional regularized moment system can be written as

$$\frac{D\mathbf{w}}{Dt} + \sum_{d=1}^D \tilde{\mathbf{A}}_M^{(d)} \frac{\partial \mathbf{w}}{\partial x_d} = 0. \quad (4.17)$$

Recalling the definition of  $\tilde{\mathbf{A}}_M^{(d)}$ , and noting the regularized collisionless moment system in one-dimensional space (4.5), we can reformulate the regularized collisionless moment systems as

$$\begin{aligned} &\frac{Df_\alpha}{Dt} + \sum_{d,k=1}^D \left( \theta_{dk} \frac{\partial f_{\alpha-e_k}}{\partial x_d} + (1 - \delta_{|\alpha|,M}) (\alpha_k + 1) \delta_{kd} \frac{\partial f_{\alpha+e_k}}{\partial x_d} \right) + \\ &\sum_{i=1}^D f_{\alpha-e_i} \frac{Du_i}{Dt} + \sum_{i,d,k=1}^D (\theta_{dk} f_{\alpha-e_i-e_k} + (1 - \delta_{|\alpha|,M}) (\alpha_k + 1) \delta_{kd} f_{\alpha-e_i+e_k}) \frac{\partial u_i}{\partial x_d} + \\ &\sum_{i,j=1}^D \frac{f_{\alpha-e_i-e_j}}{2} \frac{D\theta_{ij}}{Dt} + \sum_{i,j,d,k=1}^D \frac{1}{2} (\theta_{kd} f_{\alpha-e_i-e_j-e_k} + (1 - \delta_{|\alpha|,M}) (\alpha_k + 1) \delta_{kd} f_{\alpha-e_i-e_j+e_k}) \frac{\partial \theta_{ij}}{\partial x_d} \\ &= 0, \quad |\alpha| \leq M. \end{aligned} \quad (4.18)$$

Actually, (4.18) is obtained by using the regularization on (2.10). Since the moment system (3.7) is derived from (2.10) by eliminating the material derivatives of  $u_d$  and  $\theta_{ij}$ , there exists an invertible matrix  $\mathbf{T}(\mathbf{w})$  depending on  $\mathbf{w}$  such that the regularized moment system is identical to the following system:

$$\mathbf{T}(\mathbf{w}) \frac{D\mathbf{w}}{Dt} + \sum_{d=1}^D \mathbf{T}(\mathbf{w}) \tilde{\mathbf{A}}_M^{(d)} \frac{\partial \mathbf{w}}{\partial x_d} = 0. \quad (4.19)$$

The following theorem declares the hyperbolicity<sup>1</sup> of the multi-dimensional regularized moment system (4.17):

**Theorem 4.13.** *The regularized moment system (4.17) is hyperbolic for any admissible  $\mathbf{w}$ . Precisely, for a given unit vector  $\mathbf{n} = (n_1, \dots, n_D)$ , there exists a constant matrix  $\mathbf{Z}$  partially depending on  $\mathbf{n}$  such that*

$$\sum_{d=1}^D n_d \tilde{\mathbf{A}}_M^{(d)}(\mathbf{w}) = \mathbf{Z}^{-1} \tilde{\mathbf{A}}_M^{(1)}(\mathbf{Z}\mathbf{w}) \mathbf{Z}, \quad (4.20)$$

and the matrix is diagonalizable with eigenvalues as

$$C_{n,m} \sqrt{\mathbf{n}^T \Theta \mathbf{n}}, \quad 1 \leq n \leq m \leq M+1. \quad (4.21)$$

Actually, this theorem gives the rotation invariance of the regularized moment system (4.17) and its globally hyperbolicity. Property 1 indicates the translation invariance of the moment system, hence, it is concluded that the regularized system is Galilean invariant. If another coordinate  $(x_1^*, \dots, x_D^*)$  is adopted and the vector  $\mathbf{n}$  is along the  $x_1^*$ -axis, then the rotated moment system is equivalent to the original one. The rotation invariance is intuitive: on one hand, the moment system (3.7) is rotationally invariant, since the full  $M$ -degree polynomials are used in the truncated expansion; on the other hand, the regularization is symmetric in every direction. In the following, we will give a rigorous proof of this theorem.

Let  $\mathbf{G} = (g_{ij})_{D \times D}$  to be the rotation matrix, thus  $\mathbf{G}$  is orthogonal and its determinant is 1. We define

$$x_i^* = \sum_{j=1}^D g_{ij} x_j, \quad i = 1, \dots, D, \quad (4.22)$$

and denote by  $\rho^*$ ,  $\mathbf{u}^*$  and  $\Theta^*$  the density, macroscopic velocity and temperature tensor in the new coordinate  $\mathbf{x}^* = (x_1^*, \dots, x_D^*)$ . If we define  $\boldsymbol{\xi}^* = \mathbf{G}\boldsymbol{\xi}$ , then the orthogonality of  $\mathbf{G}$  shows

$$\rho^* = \int_{\mathbb{R}^D} f(\boldsymbol{\xi}) d\boldsymbol{\xi}^* = \int_{\mathbb{R}^D} f(\boldsymbol{\xi}) d\boldsymbol{\xi} = \rho, \quad (4.23a)$$

$$\rho^* \mathbf{u}^* = \int_{\mathbb{R}^D} \boldsymbol{\xi}^* f(\boldsymbol{\xi}) d\boldsymbol{\xi}^* = \int_{\mathbb{R}^D} \mathbf{G} \boldsymbol{\xi} f(\boldsymbol{\xi}) d\boldsymbol{\xi} = \rho \mathbf{G} \mathbf{u}, \quad (4.23b)$$

$$\rho^* \theta_{ij}^* = \int_{\mathbb{R}^D} \xi_i^* \xi_j^* f(\boldsymbol{\xi}) d\boldsymbol{\xi}^* = \int_{\mathbb{R}^D} \sum_{k,l=1}^D g_{ik} \xi_k g_{jl} \xi_l f(\boldsymbol{\xi}) d\boldsymbol{\xi} = \sum_{k,l=1}^D g_{ik} g_{jl} \theta_{kl}, \quad (4.23c)$$

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<sup>1</sup>For multi-dimensional quasi-linear systems, we refer the readers to [17] for the definition of hyperbolicity.

thus we have

$$\mathbf{u}^* = \mathbf{G}\mathbf{u}, \quad \Theta^* = \mathbf{G}\Theta\mathbf{G}^T. \quad (4.24)$$

Consider the two expansions

$$f(\boldsymbol{\xi}) = \sum_{\alpha \in \mathbb{N}^D} f_\alpha \mathcal{H}_\alpha^{[\Theta]}(\boldsymbol{\xi} - \mathbf{u}) = \sum_{\alpha \in \mathbb{N}^D} f_\alpha^* \mathcal{H}_\alpha^{[\Theta^*]}(\boldsymbol{\xi}^* - \mathbf{u}^*). \quad (4.25)$$

We have the following result.

**Lemma 4.14.** *For any  $m \in \mathbb{N}$ , there exists a group of constants  $Q_\alpha^\beta$ ,  $|\alpha| = |\beta| = m$ , such that for any  $|\alpha| = m$ ,*

$$\begin{aligned} \frac{D^* f_\alpha^*}{D^* t} + \sum_{i=1}^D f_{\alpha-e_i}^* \frac{D^* u_i^*}{D^* t} + \sum_{i,j=1}^D \frac{f_{\alpha-e_i-e_j}^*}{2} \frac{D^* \theta_{ij}^*}{D^* t} \\ = \sum_{|\beta|=|\alpha|} Q_\beta^\alpha \left( \frac{D f_\alpha}{D t} + \sum_{i=1}^D f_{\alpha-e_i} \frac{D u_i}{D t} + \sum_{i,j=1}^D \frac{f_{\alpha-e_i-e_j}}{2} \frac{D \theta_{ij}}{D t} \right), \end{aligned} \quad (4.26a)$$

$$\begin{aligned} \sum_{d,k=1}^D \theta_{dk}^* \left( \frac{\partial f_{\alpha-e_k}^*}{\partial x_d^*} + \sum_{i=1}^D f_{\alpha-e_i-e_k}^* \frac{\partial u_i^*}{\partial x_d^*} + \sum_{i,j=1}^D \frac{1}{2} f_{\alpha-e_i-e_j}^* \frac{\partial \theta_{ij}^*}{\partial x_d^*} \right) \\ = \sum_{|\beta|=|\alpha|} Q_\beta^\alpha \left( \sum_{d,k=1}^D \theta_{dk} \left( \frac{\partial f_{\alpha-e_k}}{\partial x_d} + \sum_{i=1}^D f_{\alpha-e_i-e_k} \frac{\partial u_i}{\partial x_d} + \sum_{i,j=1}^D \frac{1}{2} f_{\alpha-e_i-e_j} \frac{\partial \theta_{ij}}{\partial x_d} \right) \right), \end{aligned} \quad (4.26b)$$

$$\begin{aligned} \sum_{d=1}^D (\alpha_d + 1) \left( \frac{\partial f_{\alpha+e_d}^*}{\partial x_d^*} + \sum_{i=1}^D f_{\alpha-e_i+e_d}^* \frac{\partial u_i^*}{\partial x_d^*} + \sum_{i,j=1}^D \frac{1}{2} f_{\alpha-e_i-e_j+e_d}^* \frac{\partial \theta_{ij}^*}{\partial x_d^*} \right) \\ \sum_{|\beta|=|\alpha|} Q_\beta^\alpha \left( \sum_{d=1}^D (\alpha_d + 1) \left( \frac{\partial f_{\alpha+e_d}}{\partial x_d} + \sum_{i=1}^D f_{\alpha-e_i+e_d} \frac{\partial u_i}{\partial x_d} + \sum_{i,j=1}^D \frac{1}{2} f_{\alpha-e_i-e_j+e_d} \frac{\partial \theta_{ij}}{\partial x_d} \right) \right), \end{aligned} \quad (4.26c)$$

where  $\frac{D^* \cdot}{D^* t}$  denotes  $\frac{\partial \cdot}{\partial t} + \sum_{d=1}^D u_d^* \frac{\partial \cdot}{\partial x_d^*}$ .

*Proof.* Since  $\mathbf{u}^* = \mathbf{G}\mathbf{u}$  and  $\mathbf{x}^* = \mathbf{G}\mathbf{x}$  hold, and  $\mathbf{G}$  is an orthogonal matrix, we have

$$\sum_{d=1}^D u_d^* \frac{\partial \cdot}{\partial x_d^*} = \sum_{d=1}^D \sum_{i=1}^D g_{di} u_i \sum_{j=1}^D g_{dj} \frac{\partial \cdot}{\partial x_j} = \sum_{i,j=1}^D \delta_{ij} u_i \frac{\partial \cdot}{\partial x_j} = \sum_{d=1}^D u_d \frac{\partial \cdot}{\partial x_d}.$$

Here  $\sum_{d=1}^D g_{id} g_{jd} = \delta_{ij}$  is used in the second equality. Thus, we have

$$\frac{D \cdot}{D t} = \frac{D^* \cdot}{D^* t}.$$

Since

$$\frac{Df(\boldsymbol{\xi})}{Dt} = \sum_{\alpha \in \mathbb{N}^D} \left( \frac{Df_\alpha}{Dt} + \sum_{i=1}^D f_{\alpha-e_i} \frac{Du_i}{Dt} + \sum_{i,j=1}^D \frac{f_{\alpha-e_i-e_j}}{2} \frac{D\theta_{ij}}{Dt} \right) \mathcal{H}_\alpha^{[\Theta]}(\boldsymbol{\xi} - \mathbf{u}),$$

considering the expansion (4.25), we have

$$\begin{aligned} & \sum_{\alpha \in \mathbb{N}^D} \left( \frac{Df_\alpha}{Dt} + \sum_{i=1}^D f_{\alpha-e_i} \frac{Du_i}{Dt} + \sum_{i,j=1}^D \frac{f_{\alpha-e_i-e_j}}{2} \frac{D\theta_{ij}}{Dt} \right) \mathcal{H}_\alpha^{[\Theta]}(\boldsymbol{\xi} - \mathbf{u}) \\ &= \sum_{\alpha \in \mathbb{N}^D} \left( \frac{Df_\alpha^*}{Dt} + \sum_{i=1}^D f_{\alpha-e_i}^* \frac{Du_i^*}{Dt} + \sum_{i,j=1}^D \frac{f_{\alpha-e_i-e_j}^*}{2} \frac{D\theta_{ij}^*}{Dt} \right) \mathcal{H}_\alpha^{[\Theta^*]}(\boldsymbol{\xi}^* - \mathbf{u}^*). \end{aligned} \quad (4.27)$$

The rotation relation (A.21) of the generalized Hermite functions indicates that there exists a group of constants  $Q_\alpha^\beta$ ,  $|\alpha| = |\beta|$ , such that

$$\mathcal{H}_\alpha^{[\Theta]}(\mathbf{x}) = \sum_{|\beta|=|\alpha|} Q_\alpha^\beta \mathcal{H}_\beta^{[\mathbf{G}\Theta\mathbf{G}^T]}(\mathbf{G}\mathbf{x}), \quad (4.28)$$

and the matrix  $(Q_\alpha^\beta)_{|\alpha|=|\beta|=m}$ , formulated by collecting these constants  $Q_\alpha^\beta$ , is non-singular. Substituting (4.28) into (4.27), we obtain

$$\begin{aligned} & \sum_{\alpha \in \mathbb{N}^D} \sum_{|\beta|=|\alpha|} \left( \frac{Df_\alpha}{Dt} + \sum_{i=1}^D f_{\alpha-e_i} \frac{Du_i}{Dt} + \sum_{i,j=1}^D \frac{f_{\alpha-e_i-e_j}}{2} \frac{D\theta_{ij}}{Dt} \right) Q_\alpha^\beta \mathcal{H}_\beta^{[\Theta^*]}(\boldsymbol{\xi}^* - \mathbf{u}^*) \\ &= \sum_{\alpha \in \mathbb{N}^D} \left( \frac{Df_\alpha^*}{Dt} + \sum_{i=1}^D f_{\alpha-e_i}^* \frac{Du_i^*}{Dt} + \sum_{i,j=1}^D \frac{f_{\alpha-e_i-e_j}^*}{2} \frac{D\theta_{ij}^*}{Dt} \right) \mathcal{H}_\alpha^{[\Theta^*]}(\boldsymbol{\xi}^* - \mathbf{u}^*). \end{aligned}$$

Comparing the coefficient of  $\mathcal{H}_\alpha^{[\Theta^*]}(\boldsymbol{\xi}^* - \mathbf{u}^*)$ , we obtain (4.26a).

Analogously, since  $\Theta^* = \mathbf{G}\Theta\mathbf{G}^T$ , we have

$$\sum_{d=1}^D (\xi_d^* - u_d^*) \frac{\partial \cdot}{\partial x_d^*} = \sum_{d,i,j=1}^D g_{id}(\xi_i - u_i) g_{jd} \frac{\partial \cdot}{\partial x_j} = \sum_{d=1}^D (\xi_d - u_d) \frac{\partial \cdot}{\partial x_d},$$

and

$$\sum_{k,d=1}^D \theta_{kd}^* \frac{\partial}{\partial \xi_k^*} \frac{\partial \cdot}{\partial x_d^*} = \sum_{k,d,i,j=1}^D g_{ki} \theta_{ij} g_{dj} g_{ki} \frac{\partial}{\partial \xi_i} g_{dj} \frac{\partial \cdot}{\partial x_j} = \sum_{k,d=1}^D \theta_{kd} \frac{\partial}{\partial \xi_k} \frac{\partial \cdot}{\partial x_d}.$$

Since

$$\begin{aligned} & \sum_{k,d=1}^D \theta_{kd} \frac{\partial}{\partial \xi_k} \frac{\partial f(\boldsymbol{\xi})}{\partial x_d} = \\ & - \sum_{\alpha \in \mathbb{N}^D} \sum_{k,d=1}^D \theta_{kd} \left( \frac{\partial f_{\alpha-e_k}}{\partial x_d} + \sum_{i=1}^D f_{\alpha-e_k-e_i} \frac{\partial u_i}{\partial x_d} + \sum_{i,j=1}^D \frac{f_{\alpha-e_k-e_i-e_j}}{2} \frac{\partial \theta_{ij}}{\partial x_d} \right) \mathcal{H}_\alpha^{[\Theta]}(\boldsymbol{\xi} - \mathbf{u}), \end{aligned} \quad (4.29)$$

using the same procedure in proving (4.26a), we can have (4.26b).

For the operator  $\sum_{d=1}^D (\xi_d - u_d) \frac{\partial}{\partial x_d}$ , we have

$$\begin{aligned} & \sum_{d=1}^D (\xi_d - u_d) \frac{\partial f(\boldsymbol{\xi})}{\partial x_d} = \\ & - \sum_{\alpha \in \mathbb{N}^D} \left( \sum_{k,d=1}^D \theta_{kd} \left( \frac{\partial f_{\alpha-e_k}}{\partial x_d} + \sum_{i=1}^D f_{\alpha-e_k-e_i} \frac{\partial u_i}{\partial x_d} + \sum_{i,j=1}^D \frac{f_{\alpha-e_k-e_i-e_j}}{2} \frac{\partial \theta_{ij}}{\partial x_d} \right) \right. \\ & \left. \sum_{d=1}^D (\alpha_d + 1) \left( \frac{\partial f_{\alpha+e_d}^*}{\partial x_d^*} + \sum_{i=1}^D f_{\alpha-e_i+e_d}^* \frac{\partial u_i^*}{\partial x_d^*} + \sum_{i,j=1}^D \frac{1}{2} f_{\alpha-e_i-e_j+e_d}^* \frac{\partial \theta_{ij}^*}{\partial x_d^*} \right) \right) \mathcal{H}_\alpha^{[\Theta]}(\boldsymbol{\xi} - \mathbf{u}). \end{aligned}$$

Noting that the second line is that in (4.29), we can obtain (4.26c) by using the same procedure in proving (4.26a). This ends the proof.  $\square$

*Proof of Theorem 4.13.* Since  $\mathbf{n} = (n_1, \dots, n_D)$  is a unit vector, we let  $\mathbf{G} = (g_{ij})_{D \times D}$  be the orthogonal rotation matrix with its first row as  $(n_1, \dots, n_D)$ . With this rotation matrix, we define  $\mathbf{w}^*$  as (4.23) and (4.25). Then the relation between  $\mathbf{w}$  and  $\mathbf{w}^*$  is linear. Therefore, there exists a constant matrix  $\mathbf{Z}$  depending on  $\mathbf{G}$  such that

$$\mathbf{w}^* = \mathbf{Z}\mathbf{w},$$

and  $\mathbf{Z}$  is invertible, since  $\mathbf{w}$  can be obtained from  $\mathbf{w}^*$  by applying the rotation matrix  $\mathbf{G}^{-1}$ .

Lemma 4.14 have clearly shown that the “rotated equations”

$$\mathbf{T}(\mathbf{w}^*) \frac{D\mathbf{w}^*}{Dt} + \sum_{d=1}^D \mathbf{T}(\mathbf{w}^*) \tilde{\mathbf{A}}_M^{(d)}(\mathbf{w}^*) \frac{\partial \mathbf{w}^*}{\partial x_d^*} = 0. \quad (4.30)$$

can be deduced from (4.19) by a linear transformation. Hence, there exists a square matrix  $\mathbf{H}(\mathbf{w})$  such that

$$\mathbf{H}(\mathbf{w}) \mathbf{T}(\mathbf{w}) \frac{D\mathbf{w}}{Dt} + \sum_{d=1}^D \mathbf{H}(\mathbf{w}) \mathbf{T}(\mathbf{w}) \tilde{\mathbf{A}}_M^{(d)}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_d} = 0 \quad (4.31)$$

is identical to (4.30). Matching the terms with time derivatives, one can find  $\mathbf{H}(\mathbf{w}) = \mathbf{T}(\mathbf{w}^*) \mathbf{Z} \mathbf{T}^{-1}(\mathbf{w})$ . Thus (4.31) can be written as

$$\mathbf{T}(\mathbf{w}^*) \frac{D\mathbf{w}^*}{Dt} + \sum_{d=1}^D \mathbf{T}(\mathbf{w}^*) \mathbf{Z} \tilde{\mathbf{A}}_M^{(d)}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_d} = 0.$$

Noting  $\mathbf{x}^* = \mathbf{G}\mathbf{x}$ , we can rewrite the upper equation as

$$\mathbf{T}(\mathbf{w}^*) \frac{D\mathbf{w}^*}{Dt} + \sum_{j,d=1}^D g_{jd} \mathbf{T}(\mathbf{w}^*) \mathbf{Z} \tilde{\mathbf{A}}_M^{(d)}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_j^*} = 0.$$

Comparing with (4.30), one concludes

$$\sum_{d=1}^D g_{1d} \mathbf{T}(\mathbf{w}^*) \mathbf{Z} \tilde{\mathbf{A}}_M^{(d)}(\mathbf{w}) = \mathbf{T}(\mathbf{w}^*) \tilde{\mathbf{A}}_M^{(1)}(\mathbf{Z}\mathbf{w}) \mathbf{Z}.$$

Multiplying both sides by  $\mathbf{Z}^{-1} \mathbf{T}(\mathbf{w}^*)^{-1}$ , and noting  $g_{1d} = n_d$ , we obtain (4.20).

Since the macroscopic temperature tensor are  $\Theta^* = \mathbf{G} \Theta \mathbf{G}^T$  (see (4.23)), and particularly,  $\theta_{11}^* = \sum_{i,j=1}^D n_i n_j \theta_{ij}$ , the diagonalizability and (4.21) is instantly obtained using Theorem 4.4 and Lemma 4.8.  $\square$

## 5 Riemann Problem

Though the regularized moment system (4.17) is given by moment expansion up to an arbitrary order  $M$  thus extremely complex, the eigenvalues and eigenvectors of the coefficient matrix  $\tilde{\mathbf{A}}_M^{(d)}$  are rather organized, which makes it possible to study the structure of the elementary wave of this system with Riemann initial value, including the rarefaction wave, contact discontinuity and shock wave. Definitely, the structure of the elementary wave is fundamental for further investigation into the behavior of the solution of the system, and is instructional for studying the Godunov-type Riemann solver. The investigation below shows that the structure of the elementary wave of the Riemann problem is quite natural an extension of that of Euler equations, which indicates that the regularized moment system (4.17) is actually a very reasonable high order moment approximation of Boltzmann equations. Similarly as analyses of Euler equations (see [26]), we consider the  $x_1$ -split,  $D$ -dimensional Riemann problem as below:

$$\begin{cases} \frac{\partial \mathbf{w}}{\partial t} + \left( u_1 \mathbf{I} + \tilde{\mathbf{A}}_M \right) \frac{\partial \mathbf{w}}{\partial x_1} = 0, \\ \mathbf{w}(x_1, t = 0) = \begin{cases} \mathbf{w}_L, & \text{if } x_1 < 0, \\ \mathbf{w}_R, & \text{if } x_1 > 0, \end{cases} \end{cases} \quad (5.1)$$

where  $\tilde{\mathbf{A}}_M$  is equal to  $\tilde{\mathbf{A}}_M^{(1)}$ , and is defined in Definition 4.3.

Now let us recall the properties of  $\tilde{\mathbf{A}}_M$ . The characteristic polynomial of  $\tilde{\mathbf{A}}_M$  is  $\mathcal{P}_{D,M}(\lambda)$  defined in Lemma 4.8, thus the eigenvalues of  $\tilde{\mathbf{A}}_M$  are  $C_{i,m} \sqrt{\theta_{11}}$  (multiplicity is ignored),  $i = 1, \dots, m$ ,  $m = 1, \dots, M+1$ , if  $D \geq 2$ , and are  $C_{i,M+1} \sqrt{\theta_{11}}$ ,  $i = 1, \dots, M+1$  if  $D = 1$ . For each eigenvalue  $\lambda$  of  $\tilde{\mathbf{A}}_M$ , the corresponding eigenvector  $\mathbf{R}$  can be obtained by extending the corresponding diagonal block's eigenvector. And  $\mathbf{P} \mathbf{R}^2$  is eigenvector of  $\tilde{\mathbf{A}}_M$  for the eigenvalue  $\lambda$ . Property 4.11 indicates that  $\lambda R_0 \neq 0$  holds, if and only if  $He_{M+1}^{[\theta_{11}]}(\lambda) = 0$  and  $\lambda \neq 0$ . Therefore, for the matrix  $u_1 \mathbf{I} + \tilde{\mathbf{A}}_M$ ,

1. the characteristic polynomial is  $\mathcal{P}_{D,M}(\lambda - u_1)$ ;

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<sup>2</sup>  $\mathbf{P}$  satisfying  $\mathbf{P} \mathbf{w} = \mathbf{w}'$ .

2. the eigenvalues are  $u_1 + C_{i,m}\sqrt{\theta_{11}}$  (multiplicity is ignored),  $i = 1, \dots, m$ ,  $m = 1, \dots, M+1$ , if  $D \geq 2$ , and are  $u_1 + C_{i,M+1}\sqrt{\theta_{11}}$ ,  $i = 1, \dots, M+1$  if  $D = 1$ ;
3. for each eigenvalue  $\lambda$ , the corresponding eigenvectors are same as that of  $\tilde{\mathbf{A}}_M$ . Particularly,  $R_{e_1} = \frac{\lambda - u_1}{\rho} R_0$ ,  $R_{2e_1} = \frac{(\lambda - u_1)^2}{2} R_0$ ;
4. the eigenvalue and the corresponding eigenvector satisfy the relation:

$$(\lambda - u_1)R_0 \neq 0 \text{ holds, if and only if } He_{M+1}^{[\theta_{11}]}(\lambda - u_1) = 0 \text{ and } \lambda - u_1 \neq 0. \quad (5.2)$$

Since the eigenvalues and eigenvectors of coefficient matrix  $u_1 \mathbf{I} + \tilde{\mathbf{A}}_M$  are clarified, we can obtain the following result.

**Theorem 5.1.** *Each characteristic field of (5.1) is either genuinely nonlinear or linearly degenerate. And one characteristic field is genuinely nonlinear if and only if the eigenvalue  $\lambda = u_1 + C\sqrt{\theta_{11}}$  satisfies  $He_{M+1}^{[\theta_{11}]}(C\sqrt{\theta_{11}}) = 0$  and  $C \neq 0$ .*

*Proof.* Let  $\mathbf{R}$  be an eigenvector of  $\tilde{\mathbf{A}}'_M$  for the eigenvalue  $C\sqrt{\theta_{11}}$ , then  $\mathbf{P}\mathbf{R}$  is an eigenvector of  $u_1 \mathbf{I} + \tilde{\mathbf{A}}_M$  for the eigenvalue  $\lambda = u_1 + C\sqrt{\theta_{11}}$ . Since

$$\lambda = u_1 + C\sqrt{\frac{p_{11}}{\rho}},$$

depends only on  $\rho$ ,  $u_1$ ,  $p_{11}/2$ , we have

$$\begin{aligned} \nabla_{\mathbf{w}} \lambda \cdot \mathbf{R} &= -\frac{C\sqrt{\theta_{11}}}{2\rho} \cdot R_0 + 1 \cdot \frac{C\sqrt{\theta_{11}}}{\rho} R_0 + \frac{C}{\rho\sqrt{\theta_{11}}} \cdot \frac{C^2\theta_{11}}{2} R_0 \\ &= \frac{(C^2 + 1)\sqrt{\theta_{11}}}{2\rho} C R_0. \end{aligned}$$

(5.2) shows that:

1. If  $He_{M+1}^{[\theta_{11}]}(C\sqrt{\theta_{11}}) = 0$  and  $C \neq 0$ , then  $C\sqrt{\theta_{11}}R_0 \neq 0$ , thus  $\nabla_{\mathbf{w}} \lambda \cdot \mathbf{R} \equiv 0$ . Hence, this characteristic field is linearly degenerate.
2. If  $He_{M+1}^{[\theta_{11}]}(C\sqrt{\theta_{11}}) \neq 0$  or  $C = 0$ , then  $C\sqrt{\theta_{11}}R_0 = 0$ , thus  $\nabla_{\mathbf{w}} \lambda \cdot \mathbf{R} \neq 0$ . Hence, this characteristic field is genuinely nonlinear.

This completes the proof.  $\square$

The waves associated with  $\lambda$  satisfying  $He_{M+1}^{[\theta_{11}]}(\lambda - u_1) \neq 0$  or  $\lambda - u_1 = 0$  are *contact discontinuities*, and those associated with  $\lambda$  satisfying  $He_{M+1}^{[\theta_{11}]}(\lambda - u_1) = 0$  and  $\lambda - u_1 \neq 0$  will either be rarefaction waves or shock waves. Of course one does not know in advance what types of waves will be present in the solution of the Riemann problem. Below, we will study each type of waves separately in detail.



## 5.1 Rarefaction Waves

For the Riemann problem (5.1), if two states  $\mathbf{w}^L$  and  $\mathbf{w}^R$  are connected by a rarefaction wave associated with genuinely nonlinear characteristic field  $\mathbf{R}$ , which is a right eigenvector of  $\tilde{\mathbf{A}}_M$  corresponding to the eigenvalue  $\lambda = u_1 + C\sqrt{\theta_{11}}$  satisfying  $He_{M+1}^{[\theta_{11}]}(C\sqrt{\theta_{11}}) = 0$  and  $C \neq 0$ , then the following two conditions must be met:

1. constancy of the *generalized Riemann invariants* across the wave, which implies that the integral curve  $\tilde{\mathbf{w}}(\zeta) = (\tilde{w}_1(\zeta), \tilde{w}_2(\zeta), \dots, \tilde{w}_N(\zeta))$  in the  $N$ -dimensional phase space satisfies

$$\frac{d\tilde{\mathbf{w}}(\zeta)}{d\zeta} = \mathbf{R}(\tilde{\mathbf{w}}); \quad (5.3)$$

2. divergence of characteristics

$$\lambda^L = u_1^L + C\sqrt{\theta_{11}^L} < u_1^R + C\sqrt{\theta_{11}^R} = \lambda^R. \quad (5.4)$$

Actually, for a given point  $\mathbf{w}^0$  in the phase space, the integral curve across  $\mathbf{w}^0$  can be given. Since the results are rather tedious, we only give partial explicit expressions of the integral curve as below. For the characteristic field  $\mathbf{R}$  corresponding to the eigenvalue  $\lambda = u_1 + C\sqrt{\theta_{11}}$ ,

1. if  $R_0 \neq 0$ , we choose  $R_0 = \rho$ , then

$$R_{e_1} = C\sqrt{\theta_{11}}, \quad R_{2e_1} = \frac{C^2}{2}p_{11},$$

and then we have

$$\tilde{\rho}(\zeta) = \rho^0 \exp(\zeta), \quad (5.5a)$$

$$\tilde{u}_1(\zeta) = u_1^0 + \frac{2C}{C^2 - 1} \sqrt{\theta_{11}^0} \left( \exp\left(\frac{C^2 - 1}{2}\zeta\right) - 1 \right), \quad (5.5b)$$

$$\tilde{p}_{11} = p_{11}^0 \exp(C^2\zeta), \quad (5.5c)$$

where  $\theta_{11}^0 = p_{11}^0/\rho^0$ ;

2. if  $R_0 = 0$ , then  $R_{e_1} = R_{2e_1} = 0$ , we have

$$\tilde{\rho}(\zeta) = \rho^0, \quad \tilde{u}_1(\zeta) = u_1^0, \quad \tilde{p}_{11}(\zeta) = p_{11}^0. \quad (5.6)$$

One finds that (5.5) and (5.6) satisfy (5.3). Since for the rarefaction waves, the eigenvalue  $\lambda = u_1 + C\sqrt{\theta_{11}}$  satisfies  $He_{M+1}^{[\theta_{11}]}(C\sqrt{\theta_{11}}) = 0$  and  $C \neq 0$ , thus  $R_0 \neq 0$ , the eigenvalue of  $\tilde{\mathbf{A}}_M(\tilde{\mathbf{w}}(\zeta))$  is

$$\begin{aligned} \lambda(\tilde{\mathbf{w}}(\zeta)) &= \tilde{u}_1(\zeta) + C\sqrt{\frac{\tilde{p}_{11}(\zeta)}{\tilde{\rho}(\zeta)}} \\ &= u_1^0 + \frac{2C}{C^2 - 1} \sqrt{\theta_{11}^0} \left( \exp\left(\frac{C^2 - 1}{2}\zeta\right) - 1 \right) + C\sqrt{\frac{p_{11}^0 \exp(C^2\zeta)}{\rho^0 \exp(\zeta)}} \\ &= \lambda(\mathbf{w}^0) + 2C\frac{C^2 + 1}{C^2 - 1} \sqrt{\theta_{11}^0} \left( \exp\left(\frac{C^2 - 1}{2}\zeta\right) - 1 \right). \end{aligned}$$

It is clear that  $\frac{C^2+1}{C^2-1}\sqrt{\theta_{11}^0}\left(\exp\left(\frac{C^2-1}{2}\zeta\right)-1\right)$  has the same sign as  $\zeta$  for any  $C \in \mathbb{R}$ , hence,  $\lambda(\tilde{\mathbf{w}}) \geq \lambda(\mathbf{w}^0)$  if and only if  $C\zeta \geq 0$ . Therefore, for the rarefaction waves, noting (5.4), we have that:  $\lambda = u_1 + C\sqrt{\theta_{11}}$  satisfies  $He_{M+1}^{[\theta_{11}]}(C\sqrt{\theta_{11}}) = 0$  and  $C \neq 0$ , and

$$\begin{aligned} \text{if } C > 0, \text{ then } u_1^L &< u_1^R, \quad p_{11}^L < p_{11}^R; \\ \text{if } C < 0, \text{ then } u_1^L &< u_1^R, \quad p_{11}^L > p_{11}^R. \end{aligned}$$

## 5.2 Contact discontinuity

The proof of theorem 5.1 indicates that the contact discontinuity can be founded if and only if the eigenvector  $\mathbf{R}$  and the corresponding eigenvalue  $\lambda = u_1 + C\sqrt{\theta_{11}}$  satisfying  $CR_0 = 0$ . For a contact discontinuity, (5.3) is still valid, and the divergence of characteristics is replaced by

$$\lambda(\mathbf{w}^L) = \lambda(\mathbf{w}^R). \quad (5.7)$$

If  $C \neq 0$ , then  $R_0 = 0$ , (5.6) indicates that

$$u_1^L = u_1^R, \quad p_{11}^L = p_{11}^R.$$

If  $C = 0$ , then we can derive from both (5.5) and (5.6) that the upper equation is valid.

Summarizing the discussion above, we conclude that for a contact discontinuity,  $\lambda = u_1 + C\sqrt{\theta_{11}}$  satisfies  $He_{M+1}^{[\theta_{11}]}(C\sqrt{\theta_{11}}) \neq 0$  or  $C = 0$ , and

$$u_1^L = u_1^R, \quad p_{11}^L = p_{11}^R.$$

## 5.3 Shock waves

As is well known, the jump condition on the shock wave is sensitive to the form of the hyperbolic equations. Thus, we rewrite (5.1) in an appropriate form, before the discussion of the shock wave. However, (5.1) can not be written as conservation laws due to the regularization. Nevertheless, since the regularization only modifies the governing equations of  $f_\alpha$  with  $|\alpha| = M$ , (5.1) can still preserve the conservation of the else moments with orders from 0 to  $M - 1$ . Hence, (5.1) can be reformulated into  $\mathcal{N}_D((M - 1)e_D)$  conservation laws and  $N - \mathcal{N}_D((M - 1)e_D)$  non-conservative equations.

Let

$$\mathbf{F} = (F_0, F_{e_1}, \dots, F_{e_D}, F_{2e_1}, \dots, F_{Me_D})^T, \quad F_\alpha = \frac{1}{\alpha!} \int_{\mathbb{R}^D} \boldsymbol{\xi}^D f \, d\boldsymbol{\xi}, \quad |\alpha| \leq M, \quad (5.8)$$

where  $\boldsymbol{\xi}^\alpha = \prod_{d=1}^D \xi_d^{\alpha_d}$ , and  $F_0$  stands for  $F_\alpha|_{\alpha=0}$ . Then (5.1) can be written as

$$\begin{aligned} \frac{\partial F_\alpha}{\partial t} + (\alpha_1 + 1) \frac{\partial F_{\alpha+e_1}}{\partial x_1} &= 0, \quad |\alpha| < M, \\ \frac{\partial F_\alpha}{\partial t} + (\alpha_1 + 1) \frac{\partial F_{\alpha+e_1} - f_{\alpha+e_1}}{\partial x_1} \\ &\quad - (\alpha_1 + 1) \left( \sum_{i=1}^D f_{\alpha+e_1-e_i} \frac{\partial u_i}{\partial x_1} + \sum_{i,j=1}^D \frac{f_{\alpha+e_1-e_i-e_j}}{2\rho} \left( \frac{\partial p_{ij}}{\partial x_1} - \theta_{ij} \frac{\partial \rho}{\partial x_1} \right) \right) = 0, \\ &\quad |\alpha| = M. \end{aligned} \tag{5.9}$$

The integral relation (A.14) and the quasi-orthogonal relation (A.11) of the generalized Hermite polynomial indicate that there exists a function  $g_\alpha$  such that

$$F_\alpha = f_\alpha + g_\alpha(f_0, u_i, p_{ij}, f_\beta |_{|\beta| < |\alpha|})$$

Particularly, for  $|\alpha| = M$ , we have  $F_{\alpha+e_1} - f_{\alpha+e_1}$  only depends on  $\mathbf{F}$ , and

$$\rho = F_0, \quad u_i = F_{e_i}/\rho, \quad p_{ij} = (1 + \delta_{ij})F_{e_i+e_j} - \frac{F_{e_i}F_{e_j}}{F_0}.$$

For convenience, the quasi-linear form of (5.9) can be written as

$$\frac{\partial \mathbf{F}}{\partial t} + \boldsymbol{\Gamma}(\mathbf{F}) \frac{\partial \mathbf{F}}{\partial x_1} = 0, \tag{5.10}$$

where  $\boldsymbol{\Gamma}(\mathbf{F})$  is an  $N \times N$  matrix and depends on (5.9).

Since (5.10) is not a conservative system, we have to adopt the DLM theory [19] to study the shock wave. For a shock wave the two constant states  $\mathbf{F}^L$  and  $\mathbf{F}^R$  are connected through a single jump discontinuity in a genuinely non-linear field  $\mathbf{R}$ , which is a right eigenvector of  $\tilde{\mathbf{A}}_M$  corresponding to the eigenvalue  $\lambda = u_1 + C\sqrt{\theta_{11}}$  satisfying  $He_{M+1}^{[\theta_{11}]}(C\sqrt{\theta_{11}}) = 0$  and  $C \neq 0$ , travelling at the speed  $S$ . Then the following two conditions apply

- Generalized Rankine-Hugoniot condition:

$$\int_0^1 [S\mathbf{I} - \boldsymbol{\Gamma}(\boldsymbol{\Phi}(\nu; \mathbf{F}^L, \mathbf{F}^R))] \frac{\partial \boldsymbol{\Phi}}{\partial \nu}(\nu; \mathbf{F}^L, \mathbf{F}^R) d\nu = 0, \tag{5.11}$$

where  $\mathbf{I}$  is the  $N \times N$  identity matrix, and  $\boldsymbol{\Phi}(\nu; \mathbf{F}^L, \mathbf{F}^R)$  is a locally Lipschitz mapping satisfying

$$\boldsymbol{\Phi}(0; \mathbf{F}^L, \mathbf{F}^R) = \mathbf{F}^L, \quad \boldsymbol{\Phi}(1; \mathbf{F}^L, \mathbf{F}^R) = \mathbf{F}^R.$$

We refer the readers to [19] for details.

- Entropy condition:

$$\lambda(\mathbf{F}^L) > S > \lambda(\mathbf{F}^R). \tag{5.12}$$

For conservation laws, (5.11) is the same as the classical Rankine-Hugoniot condition, thus the first  $\mathcal{N}_D((M-1)e_D)$  rows of (5.11) are independent of  $\Phi$ , which make it possible to deduce some properties of the shock waves before specifying the form of  $\Phi$ .

Since

$$F_0 = \rho, \quad F_{e_1} = \rho u_1, \quad F_{2e_1} = \frac{1}{2} (p_{11} + \rho u_1^2),$$

the first equation and the  $(D+1)$ -th equation of (5.11) are

$$\rho^L u_1^L - \rho^R u_1^R = S(\rho^L - \rho^R), \quad (5.13)$$

$$p_{11}^L + \rho^L (u_1^L)^2 - p_{11}^R - \rho^R (u_1^R)^2 = S(\rho^L u_1^L - \rho^R u_1^R). \quad (5.14)$$

We assert that  $\rho^L \neq \rho^R$ . Otherwise, if  $\rho^L = \rho^R$ , then (5.13) indicates  $u_1^L = u_1^R$ , and (5.14) indicates  $p_{11}^L = p_{11}^R$ . Thus  $\lambda(\mathbf{F}^L) = \lambda(\mathbf{F}^R)$  holds, which contradicts (5.12). One can rewrite (5.13) as

$$S = \frac{\rho^L u_1^L - \rho^R u_1^R}{\rho^L - \rho^R}. \quad (5.15)$$

Substituting (5.15) into (5.12), and multiplying both sides with  $(\rho^L - \rho^R)^2$ , we obtain

$$\rho^L (\rho^L - \rho^R) (u_1^L - u_1^R) > C (\rho^L - \rho^R)^2 \sqrt{\theta_{11}^R}, \quad (5.16a)$$

$$\rho^R (\rho^L - \rho^R) (u_1^L - u_1^R) > C (\rho^L - \rho^R)^2 \sqrt{\theta_{11}^L}. \quad (5.16b)$$

If  $C > 0$ , (5.16a) gives

$$(\rho^L - \rho^R) (u_1^L - u_1^R) > 0. \quad (5.17)$$

Thus, we can divide both sides of (5.16) by  $(\rho^L - \rho^R) (u_1^L - u_1^R)$  to arrive

$$\frac{\rho^L}{\sqrt{\theta_{11}^R}} > \frac{C(\rho^L - \rho^R)}{u_1^L - u_1^R} > \frac{\rho^R}{\sqrt{\theta_{11}^L}},$$

from which one directly gets

$$\rho^L p_{11}^L - \rho^R p_{11}^R > 0. \quad (5.18)$$

Similarly, if  $C < 0$ , we have

$$(\rho^L - \rho^R) (u_1^L - u_1^R) < 0, \quad \rho^L p_{11}^L - \rho^R p_{11}^R < 0. \quad (5.19)$$

If  $S \neq 0$ , then (5.13) and (5.14) can be reformulated as

$$(\rho^L - \rho^R) (p_{11}^L - p_{11}^R) = \rho^L \rho^R (u_1^L - u_1^R)^2 > 0. \quad (5.20)$$

Here  $u_1^L \neq u_1^R$  is used. If  $S = 0$ , then (5.13) and (5.14) can be reformulated as

$$p_{11}^L - p_{11}^R + \frac{\rho^R}{\rho^L} (u_1^R)^2 (\rho^R - \rho^L) = 0,$$

thus we have

$$(\rho^L - \rho^R)(p_{11}^L - p_{11}^R) > 0. \quad (5.21)$$

Collecting (5.20) and (5.21), we observe that one and only one of the following two statements is valid

1.  $\rho^L > \rho^R$  and  $p_{11}^L > p_{11}^R$ ;
2.  $\rho^L < \rho^R$  and  $p_{11}^L < p_{11}^R$ .

If  $C > 0$ , (5.18) indicates that the first statement is valid. Then we can conclude  $u_1^L > u_1^R$  by (5.17). Similarly, if  $C < 0$ , the second statement is valid and  $u_1^L > u_1^R$ .

Now we summarize all the discussion on the entropy condition of the three types of elementary waves in the following theorem:

**Theorem 5.2.** *For the Riemann problem (5.1), for the wave of the family corresponding the eigenvalue  $\lambda = u_1 + C\sqrt{\theta_{11}}$  of  $\mathbf{A}_M$ , the macroscopic velocities and pressures on both sides of the wave have the relation with the type of the wave as in Table 1.*

Wave type	Eigenvalue	Velocity and Pressure
Rarefaction wave	$C > 0$	$u_1^L < u_1^R, p_{11}^L < p_{11}^R$
	$C < 0$	$u_1^L < u_1^R, p_{11}^L > p_{11}^R$
Shock wave	$C > 0$	$u_1^L > u_1^R, p^L > p^R$
	$C < 0$	$u_1^L > u_1^R, p^L < p^R$
Contact discontinuity	—	$u_1^L = u_1^R, p_{11}^L = p_{11}^R$

Table 1: The relation between the type classification of elementary wave and the eigenvalue, macroscopic velocity and pressure.

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## Appendix

### A Generalized Hermite Polynomials

To facilitate the derivation of Grad moment system [12], Grad gave a note of  $D$ -dimensional probabilists' isotropic Hermite polynomials in [11]. Maurice M. Mizrahi proposed the physicists' generalized Hermite polynomials and derived several properties in [20]. In this appendix, we will give a note on the probabilists' generalized Hermite polynomials to facilitate the derivation of the content.

## A.1 Definition and Notation

Consider the normalized  $D$ -dimensional weight function

$$w^{[\Theta]}(\mathbf{x}) = \frac{1}{\sqrt{\det(2\pi\Theta)}} \exp\left(-\frac{1}{2}\mathbf{x}^T\Theta^{-1}\mathbf{x}\right), \quad \text{such that } \int_{\mathbb{R}^D} w^{[\Theta]}(\mathbf{x}) d\mathbf{x} = 1, \quad (\text{A.1})$$

where  $\mathbf{x} \in \mathbb{R}^D$  and  $\Theta = (\theta_{ij}) \in \mathbb{R}^{D \times D}$  is a symmetrical positive definite matrix. A probabilists' generalized Hermite polynomials can be defined by:

$$He_{\alpha}^{[\Theta]}(\mathbf{x}) = \frac{(-1)^{|\alpha|}}{w^{[\Theta]}(\mathbf{x})} \frac{\partial^{\alpha}}{\partial \mathbf{x}^{\alpha}} w^{[\Theta]}(\mathbf{x}), \quad \alpha \in \mathbb{N}^D, \quad (\text{A.2})$$

where  $\alpha = (\alpha_1, \dots, \alpha_D)$  is a  $D$ -dimensional multi-index,  $|\alpha| = \sum_{d=1}^D \alpha_d$  and  $\frac{\partial^{\alpha}}{\partial \mathbf{x}^{\alpha}}$  denotes by  $\frac{\partial^{\alpha}}{\partial x_1^{\alpha_1} \dots \partial x_D^{\alpha_D}}$ . And denote the generalized Hermite functions by

$$\mathcal{H}_{\alpha}^{[\Theta]}(\mathbf{x}) = w^{[\Theta]}(\mathbf{x}) He_{\alpha}^{[\Theta]}(\mathbf{x}), \quad \alpha \in \mathbb{N}^D. \quad (\text{A.3})$$

If any component of  $\alpha$  is negative,  $He_{\alpha}^{[\Theta]}$  and  $\mathcal{H}_{\alpha}^{[\Theta]}$  are taken as zero for convenience.

Since  $\Theta$  is a symmetrical positive definite  $D \times D$  matrix,  $\Theta^{-1}$  is also a symmetrical positive definite matrix and denote  $\Theta^{-1} = (\theta^{ij})$ . Let  $\mathbf{X}$  denote  $\Theta^{-1}\mathbf{x}$ , i.e.  $X_i = \sum_{j=1}^D \theta^{ij} x_j$ . Then the corresponding transformation of the operators of partial differentiation can be written as

$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_D}\right)^T = \Theta^{-1} \left(\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_D}\right)^T. \quad (\text{A.4})$$

## A.2 Properties of generalized Hermite polynomials

1) We first give the following relationships:

$$\frac{\partial w^{[\Theta]}(\mathbf{x})}{\partial x_i} = -w^{[\Theta]}(\mathbf{x}) X_i, \quad \frac{\partial w^{[\Theta]}(\mathbf{x})}{\partial X_i} = -w^{[\Theta]}(\mathbf{x}) x_i. \quad (\text{A.5})$$

*Proof.* The first relation is obvious, and the second one can be directly derived from the first one and (A.4).  $\square$

2) The first few terms of  $He_{\alpha}^{[\Theta]}(\mathbf{x})$  are, for  $i, j, k, l = 1, \dots, D$ ,

$$He_0^{[\Theta]}(\mathbf{x}) = 1, \quad (\text{A.6a})$$

$$He_{e_i}^{[\Theta]}(\mathbf{x}) = X_i, \quad (\text{A.6b})$$

$$He_{e_i+e_j}^{[\Theta]}(\mathbf{x}) = X_i X_j - \theta^{ij}, \quad (\text{A.6c})$$

$$He_{e_i+e_j+e_k}^{[\Theta]}(\mathbf{x}) = X_i X_j X_k - \theta^{ij} X_k - \theta^{ik} X_j - \theta^{jk} X_i, \quad (\text{A.6d})$$

$$\begin{aligned} He_{e_i+e_j+e_k+e_l}^{[\Theta]}(\mathbf{x}) = & X_i X_j X_k X_l - \theta^{ij} X_k X_l - \theta^{ik} X_j X_l - \theta^{il} X_j X_k - \theta^{jk} X_i X_l \\ & - \theta^{jl} X_i X_k - \theta^{kl} X_i X_j + \theta^{ij} \theta^{kl} + \theta^{ik} \theta^{jl} + \theta^{il} \theta^{jk}. \end{aligned} \quad (\text{A.6e})$$

Here  $\theta^{ij} = \theta^{ji}$  is used.  $e_i$ ,  $i = 1, \dots, D$  is the  $D$ -dimensional unit multi-index with its  $i$ -th entry equal to 1.

3) Recurrence relation: for  $i = 1, \dots, D$ ,

$$He_{\alpha+e_i}^{[\Theta]}(\mathbf{x}) = X_i He_{\alpha}^{[\Theta]}(\mathbf{x}) - \frac{\partial He_{\alpha}^{[\Theta]}}{\partial x_i}. \quad (\text{A.7})$$

*Proof.* Considering the derivation of  $w^{[\Theta]}(\mathbf{x}) He_{\alpha}^{[\Theta]}(\mathbf{x})$  with respect to  $x_i$ ,  $i = 1, \dots, D$ ,

$$\begin{aligned} \frac{\partial w^{[\Theta]}(\mathbf{x}) He_{\alpha}^{[\Theta]}(\mathbf{x})}{\partial x_i} &= w^{[\Theta]}(\mathbf{x}) \frac{\partial He_{\alpha}^{[\Theta]}(\mathbf{x})}{\partial x_i} - X_i w^{[\Theta]}(\mathbf{x}) He_{\alpha}^{[\Theta]}(\mathbf{x}) \quad \text{using (A.5)} \\ &= (-1)^{|\alpha|} \frac{\partial^{\alpha+e_i}}{\partial \mathbf{x}^{\alpha+e_i}} w^{[\Theta]}(\mathbf{x}) = -w^{[\Theta]}(\mathbf{x}) He_{\alpha+e_i}^{[\Theta]}(\mathbf{x}), \quad \text{using (A.2)} \end{aligned}$$

and comparing the right hand sides of the two rows, we obtain (A.7).  $\square$

4) Differential Equation:

$$\frac{\partial He_{\alpha}^{[\Theta]}(\mathbf{x})}{\partial x_i} = \sum_{j=1}^D \theta^{ij} \alpha_j He_{\alpha-e_j}^{[\Theta]}(\mathbf{x}). \quad (\text{A.8})$$

*Proof.* We use mathematical induction to prove (A.8).

*Basis:* It is obvious that (A.8) holds for  $\alpha = 0$ , since  $He_0^{[\Theta]}(\mathbf{x}) = 1$ .

*Inductive step:* Assume (A.8) holds for all  $|\alpha| \leq n$ ,  $n \in \mathbb{N}$ . (A.7) indicates for all  $0 < |\alpha| \leq n+1$ ,

$$He_{\alpha}^{[\Theta]} = X_j He_{\alpha-e_j}^{[\Theta]} - \sum_{d=1}^D \theta^{jd} (\alpha_d - \delta_{jd}) He_{\alpha-e_j-e_d}^{[\Theta]},$$

where  $j \in \{1, \dots, D\}$  satisfying  $\alpha_j > 0$ . For any  $|\alpha| = n+1$ , there exists a  $j \in \{1, \dots, D\}$  such that  $\alpha_j > 0$ . Then

$$\frac{\partial He_{\alpha}^{[\Theta]}}{\partial x_i} = \frac{\partial X_j He_{\alpha-e_j}^{[\Theta]}}{\partial x_i} - \sum_{d=1}^D \theta^{jd} (\alpha_d - \delta_{jd}) \frac{\partial He_{\alpha-e_j-e_d}^{[\Theta]}}{\partial x_i}.$$

Since

$$\begin{aligned} \frac{\partial X_j He_{\alpha-e_j}^{[\Theta]}}{\partial x_i} &= \theta^{ij} He_{\alpha-e_j}^{[\Theta]} + X_j \sum_{d=1}^D \theta^{id} (\alpha_d - \delta_{jd}) He_{\alpha-e_j-e_d}^{[\Theta]} \\ &= \theta^{ij} He_{\alpha-e_j}^{[\Theta]} + \sum_{d=1}^D \theta^{id} (\alpha_d - \delta_{jd}) \left( He_{\alpha-e_d}^{[\Theta]} + \sum_{k=1}^D \theta^{jk} (\alpha_k - \delta_{jd} - \delta_{kd}) He_{\alpha-e_j-e_k-e_d}^{[\Theta]} \right) \\ &= \sum_{d=1}^D \theta^{id} \alpha_d He_{\alpha-e_d}^{[\Theta]} + \sum_{d=1}^D \theta^{jd} (\alpha_d - \delta_{jd}) \frac{\partial He_{\alpha-e_j-e_d}^{[\Theta]}}{\partial x_i}. \end{aligned}$$

Hence (A.8) holds for all  $|\alpha| = n + 1$ .

This finishes the proof.  $\square$

- 5) Recurrence relation again: Combing (A.7) and (A.8), we obtain for  $i = 1, \dots, D$ ,

$$He_{\alpha+e_i}^{[\Theta]}(\mathbf{x}) = X_i He_{\alpha}^{[\Theta]}(\mathbf{x}) - \sum_{j=1}^D \theta^{ij} \alpha_j He_{\alpha-e_j}^{[\Theta]}(\mathbf{x}). \quad (\text{A.9})$$

Furthermore, we have, for  $d = 1, \dots, D$ ,

$$x_d He_{\alpha}^{[\Theta]}(\mathbf{x}) = \sum_{j=1}^D \theta_{dj} He_{\alpha+e_j}^{[\Theta]}(\mathbf{x}) + \alpha_d He_{\alpha-e_d}^{[\Theta]}(\mathbf{x}). \quad (\text{A.10})$$

*Proof.* (A.9) is obviously holds. Since  $x_d = \sum_{i=1}^D \theta_{id} X_i$  and  $\sum_{i=1}^D \theta_{di} \theta_{ij} = \delta_{dj}$ , multiplying (A.9) by  $\theta_{id}$  and summing it by  $i$  yield (A.10).  $\square$

- 6) Quasi orthogonal relation:

$$\int_{\mathbb{R}^D} He_{\alpha}^{[\Theta]}(\mathbf{x}) He_{\beta}^{[\Theta]}(\mathbf{x}) w^{[\Theta]} d\mathbf{x} = C_{\alpha,\beta} \delta_{|\alpha|,|\beta|}, \quad (\text{A.11})$$

where  $C_{\alpha,\beta}$  is constant dependent on  $\alpha, \beta$ , and  $\Theta$ .

*Proof.* It is obvious (A.11) holds for  $\alpha = \beta = 0$  with  $C_{0,0} = 1$ . Without loss of generality, we assume  $0 < |\alpha| \geq |\beta|$  and  $\alpha_1 > 0$ . Since

$$\frac{\partial w^{[\Theta]} He_{\alpha}^{[\Theta]}}{\partial x_i} = (-1)^{|\alpha|} \frac{\partial^{\alpha+e_i}}{\partial \mathbf{x}^{\alpha+e_i}} w^{[\Theta]} = -w^{[\Theta]} He_{\alpha+e_i}^{[\Theta]} \quad (\text{A.12})$$

holds for  $i = 1, \dots, D$ , using the integration of parts on  $\int_{\mathbb{R}^D} He_{\alpha}^{[\Theta]}(\mathbf{x}) He_{\beta}^{[\Theta]}(\mathbf{x}) w^{[\Theta]} d\mathbf{x}$  yields

$$\begin{aligned} \int_{\mathbb{R}^D} He_{\alpha}^{[\Theta]} He_{\beta}^{[\Theta]} w^{[\Theta]} d\mathbf{x} &= - \int_{\mathbb{R}^{D-1}} dx_2 \cdots dx_D \int_{\mathbb{R}} He_{\beta}^{[\Theta]} d \left( w^{[\Theta]} He_{\alpha-e_1}^{[\Theta]} \right) \\ &= \int_{\mathbb{R}^D} w^{[\Theta]} He_{\alpha-e_1}^{[\Theta]} \sum_{j=1}^D \theta^{1j} \beta_j He_{\beta-e_j}^{[\Theta]} d\mathbf{x} \\ &= \sum_{j=1}^D \theta^{1j} \beta_j \int_{\mathbb{R}^D} He_{\alpha-e_1}^{[\Theta]} He_{\beta-e_j}^{[\Theta]} w^{[\Theta]} d\mathbf{x}, \end{aligned} \quad (\text{A.13})$$

It's used that  $\int_{\mathbb{R}^{D-1}} w^{[\Theta]} He_{\alpha-e_1}^{[\Theta]} He_{\beta}^{[\Theta]} dx_2 \cdots dx_D|_{-\infty}^{\infty} = 0$  holding for all  $\alpha, \beta \in \mathbb{N}^D$ .

If  $|\alpha| > |\beta|$ , repeating (A.13) till some entry of the subscript of  $He^{[\Theta]}$  negative, we can obtain  $\int_{\mathbb{R}^D} He_{\alpha}^{[\Theta]}(\mathbf{x}) He_{\beta}^{[\Theta]}(\mathbf{x}) w^{[\Theta]} d\mathbf{x} = 0$ .  $\square$



7) Integral relation: for  $\alpha, \beta \in \mathbb{N}^D$  and  $|\alpha| = |\beta|$ ,

$$\int_{\mathbb{R}^D} w^{[\Theta]} He_{\alpha}^{[\Theta]}(\mathbf{x} - \mathbf{a}) \mathbf{x}^{\beta} d\mathbf{x} = \alpha! \delta_{\alpha, \beta}, \quad (\text{A.14})$$

where  $\mathbf{x}^{\beta} = \prod_{d=1}^D x_d^{\beta_d}$ ,  $\delta_{\alpha, \beta} = \prod_{d=1}^D \delta_{\alpha_d, \beta_d}$ , and  $\mathbf{a}$  is a constant vector.

*Proof.* It is obvious that (A.14) holds for  $\alpha = 0$ . Without loss of generality, we assume  $|\alpha| > 0$ . For convenience, let  $C_{\alpha, \beta} = \int_{\mathbb{R}^D} w^{[\Theta]} He_{\alpha}^{[\Theta]}(\mathbf{x} - \mathbf{a}) \mathbf{x}^{\beta} d\mathbf{x}$ . Since  $|\beta| > 0$ , there exists an  $i \in \{1, \dots, D\}$  such that  $\beta_i > 0$ . Using the recurrence relation (A.10), we can obtain

$$\begin{aligned} C_{\alpha, \beta} &= \int_{\mathbb{R}^D} w^{[\Theta]} He_{\alpha}^{[\Theta]}(\mathbf{x} - \mathbf{a}) \mathbf{x}^{\beta - e_i} (x_i - a_i + a_i) d\mathbf{x} \\ &= a_i C_{\alpha, \beta - e_i} + \sum_{d=1}^D \theta_{ij} C_{\alpha + e_j, \beta - e_i} + \alpha_i C_{\alpha - e_i, \beta - e_i}. \end{aligned}$$

The quasi-orthogonal (A.11) of Hermite polynomials indicates that

$$C_{\alpha, \beta} = 0, \quad \text{if } |\beta| < |\alpha|.$$

Thus we have

$$C_{\alpha, \beta} = \alpha_i C_{\alpha - e_i, \beta - e_i}.$$

Since  $C_{0,0} = 1$ , using the mathematical induction on  $\alpha$ , one can prove (A.14) is valid.  $\square$

### A.3 Properties of generalized Hermite functions

1) Recursion relation: for  $i = 1, \dots, D$ ,

$$\mathcal{H}_{\alpha + e_i}^{[\Theta]}(\mathbf{x}) = X_i \mathcal{H}_{\alpha}^{[\Theta]}(\mathbf{x}) - \sum_{j=1}^D \theta^{ij} \alpha_j \mathcal{H}_{\alpha - e_j}^{[\Theta]}(\mathbf{x}), \quad (\text{A.15})$$

$$x_d \mathcal{H}_{\alpha}^{[\Theta]}(\mathbf{x}) = \sum_{j=1}^D \theta_{jd} \mathcal{H}_{\alpha + e_j}^{[\Theta]}(\mathbf{x}) + \alpha_d \mathcal{H}_{\alpha - e_d}^{[\Theta]}(\mathbf{x}). \quad (\text{A.16})$$

These two equations can be derived directly from (A.9) and (A.10), respectively.

2) Quasi-orthogonality relation:

$$\int_{\mathbb{R}^D} \mathcal{H}_{\alpha}^{[\Theta]}(\mathbf{x}) \mathcal{H}_{\beta}^{[\Theta]}(\mathbf{x}) \frac{1}{w^{[\Theta]}} d\mathbf{x} = C_{\alpha, \beta} \delta_{|\alpha|, |\beta|}, \quad (\text{A.17})$$

where  $C_{\alpha, \beta}$  is same as that in (A.11). And the equation can be obtained from (A.11) directly.

3) Differential relations:

$$\frac{\partial \mathcal{H}_\alpha^{[\Theta]}(\mathbf{x})}{\partial x_i} = -\mathcal{H}_{\alpha+e_i}^{[\Theta]}(\mathbf{x}), \quad (\text{A.18})$$

$$\frac{d\mathcal{H}_\alpha^{[\Theta(\tau)]}(\mathbf{x}(\tau))}{d\tau} = -\sum_{i=1}^D \mathcal{H}_{\alpha+e_i}^{[\Theta(\tau)]}(\mathbf{x}(\tau)) \frac{dx_i(\tau)}{d\tau} + \frac{1}{2} \sum_{i,j=1}^D \mathcal{H}_{\alpha+e_i+e_j}^{[\Theta(\tau)]}(\mathbf{x}(\tau)) \frac{d\theta_{ij}(\tau)}{d\tau}, \quad (\text{A.19})$$

*Proof.* The first relation is what (A.12) tells. For the second one, we first list some useful results in matrix calculus as following (see [22] for details). For a symmetrical positive definite matrix  $\Theta(\tau) = (\theta_{ij}(\tau)) \in \mathbb{R}^{D \times D}$ , and  $\Theta^{-1} = (\theta^{ij}) = (\theta^{(1)}, \dots, \theta^{(D)})$ ,

$$\frac{d\mathbf{x}^T \Theta^{-1} \mathbf{x}}{d\tau} = 2\mathbf{x}^T \Theta^{-1} \frac{d\mathbf{x}}{d\tau} + \mathbf{x}^T \frac{d\Theta^{-1}}{d\tau} \mathbf{x}, \quad (\text{A.20a})$$

$$\frac{d\Theta^{-1}(\tau)}{d\tau} = -\Theta^{-1} \frac{d\Theta(\tau)}{d\tau} \Theta^{-1} = -\sum_{i,j=1}^D \theta^{(i)} (\theta^{(j)})^T \frac{d\theta_{ij}}{d\tau}, \quad (\text{A.20b})$$

$$\frac{d \ln(|\Theta|)}{d\tau} = \text{trace} \left( \Theta^{-1} \frac{d\Theta}{d\tau} \right) = \sum_{i,j=1}^D \theta^{ij} \frac{d\theta_{ij}}{d\tau}. \quad (\text{A.20c})$$

Then we have the relation:

$$\begin{aligned} \frac{dw^{[\Theta(\tau)]}(\mathbf{x}(\tau))}{d\tau} &= w^{[\Theta]} \left( -\frac{1}{2} \frac{d \ln(|\Theta|)}{d\tau} - \Theta^{-1} \mathbf{x} \frac{d\mathbf{x}}{d\tau} - \frac{1}{2} \mathbf{x}^T \frac{d\Theta^{-1}}{d\tau} \mathbf{x} \right) \\ &= w^{[\Theta]} \sum_{i,j=1}^D \left( -\frac{1}{2} \theta^{ij} \frac{d\theta_{ij}}{d\tau} - \theta^{ij} x_j \frac{dx_i}{d\tau} + \frac{1}{2} \mathbf{x}^T \theta^{(i)} (\theta^{(j)})^T \mathbf{x} \frac{d\theta_{ij}}{d\tau} \right) \\ &= -w^{[\Theta]} \sum_i H e_{e_i}^{[\Theta]}(\mathbf{x}) \frac{dx_i}{d\tau} + w^{[\Theta]} \sum_{i,j=1}^D \frac{1}{2} H e_{e_i+e_j}^{[\Theta]}(\mathbf{x}) \frac{d\theta_{ij}}{d\tau}. \end{aligned}$$

Here (A.6b), (A.6c) and (A.20) are used. Since the definition of  $\mathcal{H}^{[\Theta]}$  (A.3) indicates

$$(-1)^{|\alpha|} \frac{\partial^\alpha}{\partial \mathbf{x}^\alpha} \left( \mathcal{H}_\beta^{[\Theta]} \right) = \mathcal{H}_{\alpha+\beta}^{[\Theta]},$$

we have

$$\begin{aligned} \frac{d\mathcal{H}^{[\Theta]}}{d\tau} &= (-1)^{|\alpha|} \frac{\partial^\alpha}{\partial \mathbf{x}^\alpha} \frac{dw^{[\Theta]}}{d\tau} \\ &= -(-1)^{|\alpha|} \sum_{i=1}^D \frac{dx_i}{d\tau} \frac{\partial^\alpha}{\partial \mathbf{x}^\alpha} \mathcal{H}_{e_i}^{[\Theta]} + (-1)^{|\alpha|} \frac{1}{2} \sum_{i,j=1}^D \frac{d\theta_{ij}}{d\tau} \frac{\partial^\alpha}{\partial \mathbf{x}^\alpha} \mathcal{H}_{e_i+e_j}^{[\Theta]} \\ &= -\sum_{i=1}^D \mathcal{H}_{\alpha+e_i}^{[\Theta]} \frac{dx_i}{d\tau} + \frac{1}{2} \sum_{i,j=1}^D \mathcal{H}_{\alpha+e_i+e_j} \frac{d\theta_{ij}}{d\tau}. \end{aligned}$$

□

- 4) Rotation relation: let  $\mathbf{G} = (g_{ij})_{D \times D}$  be a rotation matrix, -i.e.  $\mathbf{G}$  is orthogonal and its determinant is 1, then there exists a group of constants depending on  $\mathbf{G}$  such that

$$\mathcal{H}_\alpha^{[\Theta]}(\mathbf{x}) = \sum_{|\beta|=|\alpha|} Q_\alpha^\beta \mathcal{H}_\beta^{[\mathbf{G}\Theta\mathbf{G}^T]}(\mathbf{G}\mathbf{x}), \quad (\text{A.21})$$

and if we collect  $Q_\alpha^\beta$  as a matrix  $(Q_\alpha^\beta)_{|\alpha|=|\beta|=m}$ , then it is non-singular.

*Proof.* We define the linear space

$$\mathcal{V}_m = \left\{ p(\mathbf{x}) w^{[\Theta]} \left| \int_{\mathbb{R}^D} p(\mathbf{x}) \mathcal{H}_\beta^{[\Theta]} d\mathbf{x} = 0, \forall |\beta| \neq m, p(\mathbf{x}) \text{ is a multivariate polynomial} \right. \right\}.$$

With the quasi-orthogonal relation (A.17) of  $\mathcal{H}_\alpha^{[\Theta]}$ , it is apparent that  $\mathcal{H}_\alpha^{[\Theta]}(\mathbf{x})$  with  $|\alpha| = m$  is a basis of  $\mathcal{V}_m$ , and the dimension of  $\mathcal{V}_m$  is  $\mathcal{N}_D(m e_D) - \mathcal{N}_D((m-1)e_D)$ . Hence, we just need to prove that  $\mathcal{H}_\beta^{[\mathbf{G}\Theta\mathbf{G}^T]}(\mathbf{G}\mathbf{x})$  is also a basis of  $\mathcal{V}_m$ . Since

$$w^{[\mathbf{G}\Theta\mathbf{G}^T]}(\mathbf{G}\mathbf{x}) = \frac{1}{\sqrt{|2\pi\Theta|}} \exp\left(-\frac{1}{2}(\mathbf{G}\mathbf{x})^T (\mathbf{G}\Theta\mathbf{G}^T)^{-1} (\mathbf{G}\mathbf{x})\right) = w^{[\Theta]}(\mathbf{x}),$$

$\mathcal{H}_\alpha^{[\mathbf{G}\Theta\mathbf{G}^T]}(\mathbf{G}\mathbf{x})$  can also be defined as

$$\mathcal{H}_\alpha^{[\mathbf{G}\Theta\mathbf{G}^T]}(\mathbf{G}\mathbf{x}) = (-1)^{|\alpha|} \frac{\partial^\alpha}{\partial(\mathbf{G}\mathbf{x})} w^{[\Theta]}(\mathbf{x}).$$

This means  $\mathcal{H}_\alpha^{[\mathbf{G}\Theta\mathbf{G}^T]}(\mathbf{G}\mathbf{x})$  is a rotation of  $\mathcal{H}_\alpha^{[\Theta]}(\mathbf{x})$ , thus  $\left\{ \mathcal{H}_\alpha^{[\mathbf{G}\Theta\mathbf{G}^T]}(\mathbf{G}\mathbf{x}) \right\}_{|\alpha|=m}$  is linearly independent.

The quasi orthogonal relation (A.17) indicates that for any multi-dimensional polynomial  $p(\mathbf{x})$  with its degree less than  $|\alpha|$ ,  $\int_{\mathbb{R}^D} \mathcal{H}_\alpha^{[\mathbf{G}\Theta\mathbf{G}^T]}(\mathbf{G}\mathbf{x}) p(\mathbf{x}) d\mathbf{x} = 0$ . Hence,  $\mathcal{H}_\alpha^{[\mathbf{G}\Theta\mathbf{G}^T]}(\mathbf{G}\mathbf{x})$  is orthogonal with  $\mathcal{H}_\beta^{[\Theta]}(\mathbf{x})$ ,  $|\beta| < |\alpha|$ . Analogously,  $\mathcal{H}_\beta^{[\Theta]}(\mathbf{x})$ ,  $|\beta| > |\alpha|$  is orthogonal with  $\mathcal{H}_\alpha^{[\mathbf{G}\Theta\mathbf{G}^T]}(\mathbf{G}\mathbf{x})$ . So we have  $\mathcal{H}_\alpha^{[\mathbf{G}\Theta\mathbf{G}^T]}(\mathbf{G}\mathbf{x}) \in \mathcal{V}_{|\alpha|}$ .

In conclusion,  $\mathcal{H}_\alpha^{[\mathbf{G}\Theta\mathbf{G}^T]}(\mathbf{G}\mathbf{x})$  with  $|\alpha| = m$  is a basis of  $\mathcal{V}_m$ , thus prove the property.  $\square$

## A.4 Properties of one-dimensional case

In this subsection, we study the generalized Hermite polynomials at the case  $D = 1$ , then the matrix  $\Theta$  degenerates into a scalar  $\theta$ , and  $\Theta^{-1}$  turns to  $1/\theta$ . Particularly, if  $\theta = 1$ , the generalized Hermite polynomials  $He_n^{[\theta]}(x)$  is exactly the ordinary Hermite polynomials  $He_n(x)$ . The ordinary Hermite polynomials can be defined as

$$He_n(x) = (-1)^n \exp(x^2/2) \frac{d}{dx} \exp(-x^2/2), \quad (\text{A.22})$$

and has the properties (see [1] for details)

- 1) Parity:  $He_n(-x) = (-1)^n He_n(x)$ ;
- 2) Recursion relation:  $He_{n+1}(x) = xHe_n(x) - nHe_{n-1}(x)$ ,  $n \in \mathbb{N}^+$ ;
- 3) Orthogonality relation:  $\int_{\mathbb{R}} He_n(x) He_m(x) \exp(-x^2/2) dx = \sqrt{2\pi} m! \delta_{mn}$ ;
- 4) Differential relation:  $He_n(x)' = nHe_{n-1}(x)$ .

Since the generalized Hermite polynomials in one-dimensional case is defined as

$$He_n^{[\theta]}(x) = (-1)^n \exp(x^2/2\theta) \frac{d}{dx} \exp(-x^2/2\theta), \quad (\text{A.23})$$

hence, we can obtain

$$He^{[\Theta]}(x) = \theta^{-n/2} He_n(x/\sqrt{\theta}). \quad (\text{A.24})$$

Therefore,  $He_n^{[\theta]}(x)$  satisfies the following properties:

- 1) Parity:  $He_n^{[\theta]}(-x) = (-1)^n He_n^{[\theta]}(x)$ ;
- 2) Recurrence relation:  $He_{n+1}^{[\theta]}(x) = \frac{x}{\theta} He_n^{[\theta]}(x) - \frac{n}{\theta} He_{n-1}^{[\theta]}(x)$ ,  $n \in \mathbb{N}^+$ ;
- 3) Orthogonal relation:  $\int_{\mathbb{R}} He_n^{[\theta]}(x) He_m^{[\theta]}(x) \exp(-x^2/2) dx = \sqrt{2\pi} \frac{m!}{\theta^m} \delta_{mn}$ ;
- 4) Differential relation:  $He_n(x)' = \frac{n}{\theta} He_{n-1}(x)$ .

Next we discuss the zeros of  $He_n(x)$ . The following properties can be found in many handbooks such as [9].

**Property 5.** 1) 0 is a zero of  $He_n(x)$  if  $n$  is an odd number;

2) There are  $n$  different real zeros of  $He_n(x)$ ;

3) There is a zero of  $He_{n+1}(x)$  between any two zeros of  $He_n(x)$ ;

4) There is no same zeros of  $He_n(x)$  and  $He_{n+1}(x)$ .

Furthermore, we conjecture that there is no same non-zero zeros of  $He_n(x)$  and  $He_m(x)$  for all  $m, n \in \mathbb{N}$  and  $m \neq n$ . However, to our knowledge, no proof for it has been given. We propose it as a conjecture.

**Conjecture A.1.** For any  $m, n \in \mathbb{N}$  and  $m \neq n$ , there is no common non-zero zeros of  $He_n(x)$  and  $He_m(x)$ , -i.e.  $\nexists x \in \mathbb{R} \setminus \{0\}$ , such that  $He_n(x) = He_m(x) = 0$ .

We have verified this conjecture by computer algebra system for  $m, n \leq 1000$ .

## B Proof of Lemma 4.5

*Proof of Lemma 4.5.* Choose  $\mathbf{R} \in \mathbb{R}^M$ , such that

$$R_1 = 1, \quad R_2 = \rho\lambda, \quad R_k = \rho He_{k-1}^{[\theta_{11}]}(\lambda)/(k-1)! - f_{(k-1)e_1} - \lambda f_{(k-2)e_1}, \quad k = 2, \dots, M,$$

where  $\lambda$  is an eigenvalue of  $\tilde{A}_1$ , then we verify that  $\tilde{A}_1 \mathbf{R} = \lambda \mathbf{R}$ , which is equivalent to

$$\tilde{A}_1(i, \cdot) \mathbf{R} = \lambda R_i, \quad \text{for } i = 1, \dots, M. \quad (\text{B.1})$$

It is easy to check (B.1) holding for  $i = 1, 2, 3$ . For  $i = 4, \dots, M-1$ , since the regularization changes only the entries of the last row of  $\tilde{A}_1$ , thus (3.13) gives us the entries of  $\tilde{A}_1$  as

$$\begin{aligned} \tilde{A}_1(i, 1 : 3) &= (if_{ie_1}, ((i-1)f_{(i-1)e_1} + \theta_{11}f_{(i-3)e_1})/\rho, -2f_{(i-2)e_1}/\rho), \\ \tilde{A}_1(i, i-1 : i+1) &= (\theta_{11}, 0, i). \end{aligned}$$

Note that any entries of  $\tilde{A}_1(i, \cdot)$ , if is not given above, is zero. And some entries, which is double defined above, is the sum of the both expressions. Thus

$$\begin{aligned} \tilde{A}_1(i, \cdot) \mathbf{R} &= if_{ie_1} \cdot 1 + ((i-1)f_{(i-1)e_1} + \theta_{11}f_{(i-3)e_1})/\rho \cdot \rho\lambda \\ &\quad - 2f_{(i-2)e_1}/\rho \cdot \rho He_2^{[\theta_{11}]}(\lambda)/2! + \theta_{11} \cdot \left( \rho He_{i-2}^{[\theta_{11}]}(\lambda)/(i-2)! - f_{(i-2)e_1} - \lambda f_{(i-3)e_1} \right) \\ &\quad + i \cdot \left( \rho He_i^{[\theta_{11}]}(\lambda)/i! - f_{ie_1} - \lambda f_{(i-1)e_1} \right) \\ &= \rho\theta_{11} \frac{He_{i-2}^{[\theta_{11}]}(\lambda)}{(i-2)!} + \rho \frac{i He_i^{[\theta_{11}]}(\lambda)}{i!} - \lambda f_{(i-1)e_1} - \lambda^2 f_{(i-2)e_1} \\ &= \lambda \left( \rho He_{i-1}^{[\theta_{11}]}(\lambda)/(i-1)! - f_{(i-1)e_1} - \lambda f_{(i-2)e_1} \right) \\ &= \lambda R_i. \end{aligned}$$

Note that  $He_{n+1}^{[\theta_{11}]}(\lambda) + n He_n^{[\theta_{11}]}(\lambda) = \lambda He_n^{[\theta_{11}]}(\lambda)$  is used in the calculation above. For the case  $i = M$ , the regularization (4.4) and (3.13) gives us that

$$\begin{aligned} \tilde{A}_1(M, 1 : 3) &= (0, (-f_{(i-1)e_1} + \theta_{11}f_{(i-3)e_1})/\rho, -2f_{(i-2)e_1}/\rho), \\ \tilde{A}_1(M, M-1 : M) &= (\theta_{11}, 0). \end{aligned}$$

Similarly, any entries of  $\tilde{A}_1(i, \cdot)$ , if is not given above, is taken as zero. And some entries, which is given twice above (when  $M \leq 4$ ), is the sum of the both expressions.

Thus

$$\begin{aligned}
\tilde{A}_1(i, \cdot) \mathbf{R} &= (-f_{(i-1)e_1} + \theta_{11} f_{(i-3)e_1}) / \rho \cdot \rho \lambda \\
&\quad - 2f_{(i-2)e_1} / \rho \cdot \rho He_2^{[\theta_{11}]}(\lambda) / 2! + \theta_{11} \cdot \left( \rho He_{i-2}^{[\theta_{11}]}(\lambda) / (i-2)! - f_{(i-2)e_1} - \lambda f_{(i-3)e_1} \right) \\
&= \rho \theta_{11} \frac{He_{i-2}^{[\theta_{11}]}(\lambda)}{(i-2)!} - \lambda f_{(i-1)e_1} - \lambda^2 f_{(i-2)e_1} \\
&= \lambda \left( \rho He_{i-1}^{[\theta_{11}]}(\lambda) / (i-1)! - f_{(i-1)e_1} - \lambda f_{(i-2)e_1} \right) - \rho He_M^{[\theta_{11}]}(\lambda) / M! \\
&= \lambda R_i - \rho He_M^{[\theta_{11}]}(\lambda) / M!.
\end{aligned}$$

Hence, if  $\lambda$  satisfies  $He_M^{[\theta_{11}]}(\lambda) = 0$ , then  $(\lambda, R)$  is a pair of eigenvalue/eigenvector of  $\tilde{A}_1$ .

It is clear that any root of  $He_M^{[\theta_{11}]}(\lambda)$  is an eigenvalue of  $\tilde{A}_1$ . Since  $He_M^{[\theta_{11}]}(\lambda)$  is a monic polynomial, the characteristic polynomial of  $\tilde{A}_1$  is  $He_M^{[\theta_{11}]}(\lambda)$ . This proves the lemma.  $\square$

## C Proof of Lemma 4.10

Before we begin the proof of Lemma 4.10, we list some results on linear algebra without proof.

**Lemma C.1.** *For a  $k \times k$  block lower triangular matrix  $\mathbf{A} \in \mathbb{R}^N$  with the size of diagonal block  $n_i \times n_i$ ,  $n_1 + \dots + n_k = N$ ,  $\mathbf{r}_i$  is an eigenvector of the  $i$ -th diagonal block for the eigenvalue  $\lambda$ . If  $\lambda$  is a simple eigenvalue of  $\mathbf{A}$ , then there exists a proper prolongation of  $\mathbf{r}_i$ .*

**Lemma C.2.**  *$\mathbf{A}$  is defined the same as that in Lemma C.1, and denote  $A_{ij}$  the  $i$ -th row,  $j$ -th column block of  $\mathbf{A}$ . Each diagonal block of  $\mathbf{A}$  is diagonalizable with real eigenvalues.  $\mathbf{r}_i$  is an eigenvector of  $A_{ii}$  for the eigenvalue  $\lambda$ .  $\lambda$  is an eigenvalue of  $A_{jj}$ ,  $j \neq i$ , and is not an eigenvalue of any other diagonal block of  $\mathbf{A}$ . If there exists a proper prolongation to the matrix*

$$\begin{pmatrix} A_{ii} & 0 \\ A_{ij} & A_{jj} \end{pmatrix}, \text{ if } i < j, \text{ or } \begin{pmatrix} A_{jj} & 0 \\ A_{ji} & A_{ii} \end{pmatrix}, \text{ if } i > j,$$

*then there exists a proper prolongation of  $\mathbf{r}_i$  to the matrix  $\mathbf{A}$ .*

*Proof of the Lemma 4.10.* The case  $D = 1$  has been proved in [5], here we just consider the case  $D \geq 2$ . Define  $\mathbf{R}_\alpha = (R_{\alpha, \beta})^T$ , where the order of  $R_{\alpha, \beta}$  in  $\mathbf{R}_\alpha$  is the lexicographic order of  $(\alpha_2, \dots, \alpha_D, \alpha_1)$ , same as that in  $\mathbf{w}'$ . Similarly, define  $\mathbf{r}_\alpha = (r_{\alpha, 1}, \dots, r_{\alpha, M+1-|\hat{\alpha}|})^T$ . If  $\mathbf{R}_\alpha$  is an prolongation of  $\mathbf{r}_\alpha$ , then

$$R_{\alpha, \beta} = r_{\alpha, \beta_1+1} \text{ with } \hat{\beta} = \hat{\alpha}. \quad (\text{C.1})$$

If  $\mathbf{R}_\alpha$  is an eigenvector of  $\tilde{\mathbf{A}}'_M$  for the eigenvalue  $\lambda$ , then (4.5) indicates  $R_{\alpha,\beta}$  satisfying: for  $|\beta| \leq M$ ,

$$\rho R_{\alpha,e_1} = \lambda R_{\alpha,0}, \quad (\text{C.2a})$$

$$\frac{1}{\rho}(1 + \delta_{1i})R_{\alpha,e_1+e_i} = \lambda R_{\alpha,e_i}, \quad (\text{C.2b})$$

$$p_{ij}R_{\alpha,e_1} + p_{1i}R_{\alpha,e_j} + p_{1j}R_{\alpha,e_i} + (e_i + e_j + e_1)!R_{\alpha,e_i+e_j+e_1} = \lambda(1 + \delta_{ij})R_{\alpha,e_i+e_j}, \quad (\text{C.2c})$$

$$\begin{aligned} & \sum_{k=1}^D \theta_{1k} R_{\alpha,\beta-e_k} + (1 - \delta_{|\beta|,M})(\beta_1 + 1)R_{\alpha,\beta+e_1} \\ & + \sum_{i,j=1}^D \frac{\tilde{C}_{ij}(\beta)}{2\rho} ((1 + \delta_{ij})R_{\alpha,e_i+e_j} - \theta_{ij}R_{\alpha,0}) + \sum_{i=1}^D (1 - \delta_{|\beta|,M})(\beta_1 + 1)f_{\beta-e_i+e_1}R_{\alpha,e_i} \\ & - \sum_{i=1}^D \frac{f_{\beta-e_i}}{\rho}(1 + \delta_{1i})R_{\alpha,e_1+e_i} - \sum_{i,j=1}^D \frac{(e_i + e_j + e_1)!}{2} \frac{f_{\beta-e_i-e_j}}{\rho} R_{\alpha,e_i+e_j+e_1} = \lambda R_{\alpha,\beta}, \quad |\beta| \geq 3, \end{aligned} \quad (\text{C.2d})$$

where  $\tilde{C}_{ij}(\beta)$  is defined in (4.6). We need to verify  $\mathbf{R}_{\alpha,\beta}$  satisfying (C.1) and (C.2) only, which are checked case by case below.

- 1)  $\lambda \neq 0$ ;  $D = 2$ , or  $D \geq 3$ , and  $\lambda$  satisfying  $He_{M+1}^{[\theta_{11}]}(\lambda) = 0$ . If  $D = 2$ , the characteristic polynomial of  $\tilde{\mathbf{A}}'_M$  is  $\prod_{m=1}^{M+1} He_m^{[\theta_{11}]}(\lambda)$ . Conjecture 1.1 indicates that each nonzero eigenvalue of  $\tilde{\mathbf{A}}'_M$  is a simple eigenvalue. If  $D \geq 3$ , and  $\lambda$  satisfying  $\lambda \neq 0$  and  $He_{M+1}^{[\theta_{11}]}(\lambda) = 0$ ,  $\lambda$  is a simple eigenvalue of  $\tilde{\mathbf{A}}'_M$ . By Lemma C.1, there exists a proper prolongation of each eigenvector of  $\tilde{\mathbf{A}}'_M$  associated to these eigenvalues.
- 2)  $\lambda \neq 0$ ,  $D \geq 3$  and  $\lambda$  satisfying  $He_{M+1}^{[\theta_{11}]}(\lambda) \neq 0$ . This case is corresponding to  $|\hat{\alpha}| \geq 1$ . Let  $\lambda$  be the  $\alpha_1$ -th eigenvalue  $\tilde{A}_{\hat{\alpha}}$ , then the corresponding eigenvector is  $\mathbf{r}_\alpha$ .

Conjecture 1.1 indicates that  $\lambda$  is an eigenvalue of  $\tilde{A}_{\hat{\beta}}$ ,  $|\hat{\beta}| = |\hat{\alpha}|$ , and is not for any  $\tilde{A}_{\hat{\beta}}$ ,  $|\hat{\beta}| \neq |\hat{\alpha}|$ . Here we first prolongate the eigenvector  $\mathbf{r}_\alpha$  to the diagonal block of  $\tilde{\mathbf{A}}'_M$ , containing all  $\tilde{A}_{\hat{\beta}}$ ,  $|\hat{\beta}| = |\hat{\alpha}|$  (for convenience, denote the diagonal block by  $\mathbf{B}$ ), then use Lemma C.2 to obtain a proper prolongation of  $\mathbf{r}_\alpha$ .

Actually, observing (4.5), we find that the equation, including the term  $\frac{Dw_\alpha}{Dt}$ , does not depend on  $w_\beta$ ,  $|\beta| = |\alpha|$ , which implies that  $\mathbf{B}$  is a block diagonal matrix, and particularly, each diagonal block is  $\tilde{A}_{\hat{\alpha}}$ . Hence, let  $R_{\alpha,\beta} = r_{\alpha,\beta_1+1}$  with  $\hat{\beta} = \hat{\alpha}$ , and  $R_{\alpha,\beta} = 0$  with  $\hat{\beta} \neq \hat{\alpha}$  and  $|\hat{\beta}| = |\hat{\alpha}|$ , then  $(R_{\alpha,\beta})|_{|\hat{\beta}|=|\hat{\alpha}|}$  is a prolongation of  $\mathbf{r}_\alpha$  to the matrix  $\mathbf{B}$ . Obviously,  $(R_{\alpha,\beta})|_{|\hat{\beta}|=|\hat{\alpha}|}$  is a proper prolongation of  $\mathbf{r}_\alpha$  to the matrix  $\mathbf{B}$ . With Lemma C.2, the conclusion is validated.

3)  $\lambda = 0$ . Since Hermite polynomial  $He_n(x)$  is odd function if  $n$  is odd,  $\lambda = 0$  is multi-eigenvalue of  $\tilde{\mathbf{A}}'_M$ . We have to check it in cases:

(a) Case  $|\hat{\alpha}| = 0$ . If  $\lambda = 0$  is an eigenvalue of  $\tilde{\hat{A}}_0$ , then  $M$  is even. Let

$$\begin{aligned} R_{\alpha,0} &= \rho, \quad R_{\alpha,\beta} = 0, \quad |\beta| = 1, \\ R_{\alpha,\beta} &= -\frac{1}{\beta_1} \sum_{k=1}^D \theta_{1k} (R_{\alpha,\beta-e_1-e_k} - G(\beta - e_1 - e_k)) + G(\beta), \quad |\beta| > 1, \end{aligned} \quad (\text{C.3})$$

where  $G(\beta) = \sum_{i,j=1}^D \theta_{ij} f_{\beta-e_i-e_j}$ , and  $(\cdot)_\beta$  is taken as zero if any entry of  $\beta$  is negative. Particularly,  $R_{\alpha,\beta} = 0$ , if  $|\beta| = 1, 2, 3$ , and

$$R_{\alpha,\beta} = G(\beta), \text{ if } |\beta| \text{ is odd.} \quad (\text{C.4})$$

Let  $\beta = me_1$ ,  $m = 0, \dots, M$ , it can be derived that

$$\begin{aligned} R_{\alpha,0} &= \rho, \quad R_{\alpha,e_1} = R_{\alpha,2e_1} = 0, \\ R_{\alpha,me_1} - f_{(m-2)e_1} \theta_{11} &= -\frac{1}{m} \theta_{11} (R_{\alpha,(m-2)e_1} - f_{(m-4)e_1} \theta_{11}). \end{aligned}$$

Using the recurrence relation of  $He^{[\theta_{11}]}(\lambda)$  with  $\lambda = 0$ , we find that  $R_{\alpha,me_1} = r_{\alpha,m+1}$ , where  $r_{\alpha,m+1}$  is same as that defined in (3.10) with  $\lambda = 0$ .

Then we verify that  $R_{\alpha,\beta}$  satisfies (C.2). Notice  $R_{\alpha,\beta} = 0$ ,  $|\beta| = 1, 2, 3$ , thus (C.2a), (C.2b) and (C.2c) holds, and (C.2d) degenerates into, for  $3 \leq |\beta| \leq M$ ,

$$\sum_{k=1}^D \theta_{1k} R_{\alpha,\beta-e_k} + (1 - \delta_{|\beta|,M})(\beta_1 + 1) R_{\alpha,\beta+e_1} - \sum_{i,j=1}^D \frac{\tilde{C}_{ij}(\beta)}{2\rho} \theta_{ij} R_{\alpha,0} = 0.$$

For  $|\beta| < M$ , since  $R_{\alpha,\beta-e_k} = 0$  holds for  $|\beta| = 3$ ,  $k = 1, \dots, D$ , the equation above is exactly what (C.3) tells. For  $|\beta| = M$ , since  $M$  is even,  $|\beta - e_k|$ ,  $k = 1, \dots, D$ , is odd, and this equation can be simply derived using (C.4).

(b) Case  $|\hat{\alpha}| = 1$ . If  $\lambda = 0$  is an eigenvalue of  $\tilde{\hat{A}}_1$ , then  $M$  is odd. Let  $\hat{e}_d = \hat{\alpha}$  and

$$\begin{aligned} R_{\alpha,0} &= 0, \quad R_{\alpha,e_d} = 1, \quad R_{\alpha,\beta} = 0, \quad |\beta| = 1 \text{ and } \beta \neq e_d, \\ R_{\alpha,\beta} &= -\frac{1}{\beta_1} \sum_{k=1}^D \theta_{1k} (R_{\alpha,\beta-e_1-e_k} - G(\beta - e_1 - e_k)) + G(\beta), \quad |\beta| > 1, \end{aligned} \quad (\text{C.5})$$



where  $G(\beta) = f_{\beta-e_d}$ , and  $(\cdot)_\beta$  is taken as zero if any entry of  $\beta$  is negative. Particularly,  $R_{\alpha, me_1} = 0$ ,  $m = 0, \dots, M$ , and

$$R_{\alpha, \beta} = G(\beta), \text{ if } |\beta| \text{ is even.} \quad (\text{C.6})$$

Let  $\beta = e_d + me_1$ ,  $m = 0, \dots, M-1$ , it is derived that

$$\begin{aligned} R_{\alpha, e_d} &= 1, \quad R_{\alpha, e_d+e_1} = 1, \quad R_{\alpha, e_d+2e_1} = 0, \\ R_{\alpha, e_d+me_1} - G(e_d + me_1) &= -\frac{1}{m} \theta_{11}(R_{\alpha, e_d+(m-2)e_1} - G(e_d + (m-2)e_1)). \end{aligned}$$

Using the recurrence relation of  $He^{[\theta_{11}]}(\lambda)$  with  $\lambda = 0$ , one finds that  $R_{\alpha, e_d+me_1} = r_{\alpha, m+1}$ , where  $r_{\alpha, m+1}$  is the same as that defined in Lemma 4.5 with  $\lambda = 0$ .

Then we verify that  $R_{\alpha, \beta}$  satisfies (C.2). It is clear that (C.2a), (C.2b) and (C.2c) holds. Meanwhile (C.2d) degenerates into, for  $3 \leq |\beta| \leq M$ ,

$$\begin{aligned} &\sum_{k=1}^D \theta_{1k} R_{\alpha, \beta-e_k} + (1 - \delta_{|\beta|, M})(\beta_1 + 1) R_{\alpha, \beta+e_1} \\ &+ (1 - \delta_{|\beta|, M})(\beta_1 + 1) f_{\beta+e_1-e_d} - \sum_{i,j=1}^D \frac{(e_i + e_j + e_1)!}{2} \frac{f_{\beta-e_i-e_j}}{2} R_{\alpha, e_1+e_i+e_j} = 0. \end{aligned}$$

To verify this relation is rather tedious but not complex, and we have to examine several cases for  $R_{\alpha, e_1+e_i+e_j}$ . Here we give the idea briefly. First, using (C.5) to eliminate  $R_{\alpha, e_1+e_i+e_j}$ . For  $|\beta| < M$ , one get this equation is what (C.5) tells. For  $|\beta| = M$ , since  $M$  is odd,  $|\beta - e_k|$ ,  $k = 1, \dots, D$ , is even, and then this equation is simply derived using (C.6).

Furthermore, the construction of  $\mathbf{R}_\alpha$  shows the prolongation is proper.

- (c) Case  $|\hat{\alpha}| = 2$ . If  $\lambda = 0$  is an eigenvalue of  $\tilde{A}_2$ , then  $M$  is even. Let  $\hat{\gamma} = \hat{\alpha}$ ,  $\gamma_1 = 0$ , and

$$\begin{aligned} R_{\alpha, \beta} &= 0, \quad |\beta| \leq 2, \beta \neq \gamma, \quad R_{\alpha, \gamma} = 1, \\ R_{\alpha, \beta} &= -\frac{1}{\beta_1} \sum_{k=1}^D \theta_{1k} (R_{\alpha, \beta-e_1-e_k} - G(\beta - e_1 - e_k)) + G(\beta), \quad |\beta| > |\gamma|, \end{aligned} \quad (\text{C.7})$$

where  $G(\beta) = \frac{f_{\beta-\gamma}}{\rho}$ , and  $(\cdot)_\beta$  is taken as zero if any entry of  $\beta$  is negative.

Particularly,  $R_{\alpha, \beta} = 0$ ,  $|\hat{\beta}| < 2$ , and

$$R_{\alpha, \beta} = G(\beta), \text{ if } |\beta| \text{ is odd,} \quad (\text{C.8})$$

$$R_{\alpha, \beta} = 0, \quad |\beta| = 3.$$

Let  $\beta = \gamma + me_1$ ,  $m = 0, \dots, M-2$ , it can be derived that

$$\begin{aligned} R_{\alpha, \gamma} &= 1, \quad R_{\alpha, \gamma+e_1} = 0, \\ R_{\alpha, \gamma+me_1} - G(\gamma + me_1) &= -\frac{1}{m}\theta_{11}(R_{\alpha, \gamma+(m-2)e_1} - G(\gamma + (m-2)e_1)). \end{aligned}$$

Using the recurrence relation of  $He^{[\theta_{11}]}(\lambda)$  with  $\lambda = 0$ , we can check that  $R_{\alpha, \gamma+me_1} = r_{\alpha, m+1}$ , where  $r_{\alpha, m+1}$  is the same as that defined in Lemma 4.6 with  $\lambda = 0$ .

Then we verify that  $R_{\alpha, \beta}$  satisfies (C.2). Both the left hand sides and right hand sides of (C.2a), (C.2b) and (C.2c) are zero, so these equations hold. (C.2d) degenerates into, for  $3 \leq |\beta| \leq M$ ,

$$\begin{aligned} \sum_{k=1}^D \theta_{1k} R_{\alpha, \beta-e_k} + (1 - \delta_{|\beta|, M})(\beta_1 + 1)R_{\alpha, \beta+e_1} \\ + \frac{1}{\rho} \left( \sum_{k=1}^D \theta_{1k} f_{\beta-\gamma-e_k} + (1 - \delta_{|\beta|, M})(1 + \beta_1)f_{\beta+e_1-\gamma} \right) = 0. \end{aligned}$$

For  $|\beta| < M$ , the above equation is given by (C.7). For  $|\beta| = M$ , since  $M$  is even,  $|\beta - e_k|$ ,  $k = 1, \dots, D$ , is odd, and then this equation can be simply derived using (C.8).

Furthermore, the construction of  $\mathbf{R}_\alpha$  shows the prolongation is proper.

- (d) Case  $n = |\hat{\alpha}| \geq 3$ . If  $\lambda = 0$  is an eigenvalue of  $\tilde{A}_{\hat{\alpha}}$ , then  $M+1-n$  is odd. Let  $\hat{\gamma} = \hat{\alpha}$ ,  $\gamma_1 = 0$ , and

$$\begin{aligned} R_{\alpha, \beta} &= 0, \quad |\beta| \leq n, \beta \neq \gamma, \quad R_{\alpha, \gamma} = 1, \\ R_{\alpha, \beta} &= -\frac{1}{\beta_1} \sum_{k=1}^D \theta_{1k} R_{\alpha, \beta-e_1-e_k}, \quad |\beta| > |\gamma|, \end{aligned} \quad (\text{C.9})$$

where  $(\cdot)_\beta$  is taken as zero if any entry of  $\beta$  is negative. Particularly,  $R_{\alpha, \beta} = 0$ ,  $|\hat{\beta}| < n$ , and

$$R_{\alpha, \beta} = 0, \text{ if } M+1-|\beta| \text{ is even.} \quad (\text{C.10})$$

Let  $\beta = \gamma + me_1$ ,  $m = 0, \dots, M+1-n$ , it can be derived that

$$R_{\alpha, \gamma} = 1, \quad R_{\alpha, \gamma+e_1} = 0, \quad R_{\alpha, \gamma+me_1} = -\frac{1}{m}\theta_{11}R_{\alpha, \gamma+(m-2)e_1}.$$

Using the recurrence relation of  $He^{[\theta_{11}]}(\lambda)$  with  $\lambda = 0$ , we can check that  $R_{\alpha, \gamma+me_1} = r_{\alpha, m+1}$ , where  $r_{\alpha, m+1}$  is same as that defined in Lemma 4.7 with  $\lambda = 0$ .

Next we verify that  $R_{\alpha, \beta}$  satisfies (C.2). Both the left hand sides and right hand sides of (C.2a), (C.2b) and (C.2c) are zero, so these equations

hold. (C.2d) degenerates into, for  $3 \leq |\beta| \leq M$ ,

$$\sum_{k=1}^D \theta_{1k} R_{\alpha, \beta - e_k} + (1 - \delta_{|\beta|, M})(\beta_1 + 1) R_{\alpha, \beta + e_1} = 0.$$

For  $|\beta| < n$ , both the left hand side and the right hand side are zero, so the equation holds. For  $n \leq |\beta| < M$ , this equation is exactly what (C.7) tells. For  $|\beta| = M$ , since  $M+1-n$  is even,  $M+1-|\beta-e_k|$ ,  $k = 1, \dots, D$  is odd, then the above equation can be simply derived using (C.10).

Furthermore, the construction of  $\mathbf{R}_\alpha$  shows the prolongation is proper.

All the cases discussion above tells us that each eigenvector of a diagonal block for the eigenvalue  $\lambda = 0$  can be prolongate to an eigenvector of  $\tilde{\mathbf{A}}'_M$ , and the prolongation is proper.

Collecting all the case above, we conclude that each eigenvector of each diagonal block of  $\tilde{\mathbf{A}}'_M$  can be prolongate to an eigenvector of  $\tilde{\mathbf{A}}'_M$ , which proves the Lemma.  $\square$

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